On the Singularity Set of Lorentzian Almost Einstein Structures

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INTRODUCTION

CONFORMALLY COMPLETED EINSTEIN MANIFOLDS

The method of conformal compactification goes back to the 1960's, when E. Newman and R. Penrose formalised the analysis of asymptotic behaviour of spacetimes in general relativity [NP62, Pen63, Pen65]. By attaching a non-physical conformal boundary to the spacetime they developed a method to express for example the asymptotic behaviour of the Riemann tensor, asymptotic flatness or the gravitational energy carried away by gravitational waves in terms of a non-physical metric that also is defined at the new boundary. The asymptotic behaviour of the physical quantities then can be characterised by the properties of the non-physical metric close to the boundary. A conformal compactification of a manifold $(\tilde{M}, \tilde{g})$ usually is understood as an embedding $\imath : \tilde{M} \to M$ into a bigger manifold (M, g) of the same dimension with boundary together with a smooth map $\sigma \in C^\infty(M)$, such that $\imath_* \tilde{g} = \sigma^{-2} g$, the requirement that the topological boundary $\Sigma = \partial \imath(\tilde{M})$ of the embedding is given by the zero set of σ and the requirement on $d\sigma$ not to vanish at the boundary. This picture is closely related to conformal geometry, since any conformal change $g \to e^{2\theta} g$ with a smooth map $\theta \in C^\infty(M)$ provides another compactification if the boundary defining function is changed by $\sigma \to e^\theta \sigma$. Hence if there is one compactification (M, g) of a spacetime, only the conformal class $[\gamma]$ of the induced metric γ on Σ is uniquely determined by the metric $\tilde{g}$. The class $[\gamma]$ is called conformal infinity.

At the focus of this thesis are Einstein manifolds, i.e. manifolds $(\tilde{M}, \tilde{g})$ with a Ricci tensor that is a multiple of the metric. Such metrics naturally have constant scalar curvature and are closely related to vacuum spacetimes. The latter are manifolds with a metric that solves Einstein's vacuum field equations with cosmological term $\mathrm{Ric}[\tilde{g}] - \frac{\tau^{\tilde{g}}}{2}\tilde{g} + \Lambda \tilde{g} = 0$, where $\tau^{\tilde{g}}$ is the scalar curvature of $\tilde{g}$. Such spacetimes are in fact Einstein manifolds. Einstein manifolds or vacuum spacetimes and their compactifications have been and are a rich topic of research in general relativity and pure mathematics.

There is a set of interesting questions concerning Einstein metrics. One field of interest is that on methods of attaching a most natural boundary to an Einstein manifold. This has for example be considered by G. Kronheimer and R. Penrose, who used the causal structure of a Lorentzian manifold to attach "ideal points" to it [GKP72]. In case where the Einstein manifold is asymptotically flat, this ideal points can be interpreted as conformal boundary of the manifold. B.G. Schmidt used a conformal or projective structure to define a natural boundary, called the b-boundary [Sch74, Sch73] and C. Frances considered the existence of an embedding of a pseudo-Riemannian manifold as open subset into a bigger pseudo-Riemannian manifold of same index [Fra08]. On the other hand one could start with the tuple (M, Σ) and a conformal class $[\gamma]$ on Σ. In this case one could face the problem of existence and uniqueness of an Einstein metric $\tilde{g}$ on $\tilde{M} = M \setminus \Sigma$, such that $[\gamma]$ is its conformal infinity. More generally one could start with a pseudo-Riemannian manifold (M, g) and ask whether there exists a defining function σ, such that $\sigma^{-2} g$ is Einstein outside its singularity set $\Sigma = \sigma^{-1}(0)$. If Einstein manifolds are considered in Lorentzian signature it is often useful to drop the requirement on the ambient manifold (M, g) to be compact and use the term of conformally completed Einstein manifold in this case. Therefore this thesis will focus on conformally completable Einstein manifolds in Lorentzian signature $(- + \cdots +)$.

A generalisation of conformally completable Einstein manifolds appeared when T.N. Bailey, M.G. Eastwood and R. Gover reintroduced the tractor-bundle with its tractor connection [BEG94]. The construction is based on a vector bundle found by T.Y. Thomas [Tho26] to associate conformal connections and curvature quantities to conformal structures. A parallel section in the tractor bundle can be related to an Einstein metric in the underlying conformal structure. Due to

R. Gover such conformal structures are called almost Einstein [Gov05]. Equivalently a structure (M, g, σ) is called almost Einstein, if it has a vanishing almost Einstein tensor $A[g, \sigma]$, defined by

$$A[g, \sigma] = \mathrm{Hess}^g \sigma + \sigma \mathrm{P}^g + \rho g,$$

where P^g is the Schouten tensor and $\rho = -\frac{1}{n} \mathrm{tr}^g(\mathrm{Hess}^g \sigma + \sigma \mathrm{P}^g)$. Almost Einstein structures are generalisations of conformally completable Einstein manifolds, since $\tilde{g} := \sigma^{-2} g$ is an Einstein metric away from the singularity set $\Sigma = \sigma^{-1}(0)$. The main difference to a conformal compactification as introduced by Penrose is that of M not necessarily being a manifold with boundary.

The conformal boundary in general relativity is decomposed with respect to the causal character of the gradient of the boundary defining function. The different parts are denoted spacelike, timelike and null infinity. We will recover this decomposition in the setting of almost Einstein structures and analyse its local topology (Propositions 5.1.1 and 5.1.12). This also complements the results gained by Gover for Riemannian manifolds in the Lorentzian setting. Moreover this provides a more concrete point of view towards the curved orbit decomposition found in [ČGH14] by A. Čap, A.R. Gover and M. Hammerl. In case of almost Einstein structures the curved orbit desomposition of M is a decomposition into parts that are conformally equivalent to Einstein manifolds, smoothly embedded hypersurfaces and isolated points. By showing that the isolated points and the hypersurfaces belong to the same quadric (Proposition 5.1.1) we give another explanation, why non-emptiness of the set of isolated points requires non-emptiness of the set of hypersurfaces in a Lorentzian setting.

In Riemannian signature the gradient vector field $\mathrm{grad}^g \sigma$ is orthogonal and transversal to the singularity set, which turns it into an umbilic hypersurface or a (possibly empty) set of isolated points [Gov05]. This does not longer hold true in Lorentzian signature, where the singularity set no longer needs to be a submanifold. In addition the gradient vector field is tangent to the singularity set or vanishes. We show that by removing a set of isolated points from the singularity set, the remaining part becomes a hypersurface and $\mathrm{grad}^g \sigma$ a complete tangent vector field on it (Proposition 5.1.18).

CHARACTERISTIC CAUCHY PROBLEM

The introduction of the ambient construction by C. Fefferman and C.R. Graham [FG85, FG12] was a major step in the treatment of the existence and uniqueness problem for conformally completed Einstein manifolds with a given conformal structure at the boundary. Starting with a conformal structure $(\Sigma, [\gamma])$ they constructed a generalised Poincaré Einstein metric g^+ on a thickening $\Sigma \times (0, 1]$, which has $[\gamma]$ as its conformal infinity and which is normalised, such that $\mathrm{Ric}[g^+] = \pm n g^+$. In addition $\sigma^2 g^+$ has a continuation to $\Sigma \times [0, 1]$ and can be written as $\sigma^2 g^+ = d\sigma^2 + g_\sigma$. At least if the dimension of the conformal structure is odd, the family of metrics g_σ on M is given by a formal power series, the Fefferman-Graham expansion. If the dimension of Σ is even, the continuation of the formal expansion past a critical order presumes the vanishing of the obstruction tensor $\mathcal{O}[\gamma]$ [GH05]. In fact, there exists a vast literature devoted to the convergence and regularity of that power series. For example, S. Kichenassamy showed the existence of the formal expansion in the case of real-analytic boundary data and by allowing logarithmic terms in the expansion he achieved a generalisation of the existence result to even dimensions [Kic04, Kic07].

The ambient metric construction and with it the obstruction tensor turned out to be powerful tools to analyse the Cauchy problem for conformally compactified Einstein manifolds with data prescribed at the conformal boundary. The equations provided by the vanishing of the almost Einstein tensor are equivalent to the requirement of $\tilde{g} = \sigma^{-2} g$ to be an Einstein metric. Unfortunately the system degenerates at the points where σ vanishes. This problem can be avoided by considering the obstruction tensor instead of the almost Einstein tensor. In even dimension its vanishing is a non-degenerate conformally covariant obstruction to the existence of an Einstein metric in the conformal class, at least in a dense set [FG85]. Since not every metric with vanishing obstruction tensor is conformally equivalent to an Einstein metric, this has to be treated by prescribing sufficient Cauchy data. In a Riemannian setting M.T. Anderson developed an extensive theory for boundary regularity for conformally compact Einstein metrics [And10], which is completed by comprehensive existence an uniqueness results in dimension 4 [And08]. The results are partially based on the usage of the conformally covariant Bach tensor, which is the obstruction tensor in dimension 4. Regularity and existence results in higher dimension have for example been gained by P.T. Chruściel, E. Delay, C.R. Graham, J.M. Lee and D.N. Skinner [GL91, CDLS05, Lee06].

In Lorentzian signature one basically has 3 types of conformal boundaries for compactifiable Einstein manifolds. The induced bilinear form γ can be either a Riemannian metric, a Lorentzian metric or degenerate, depending on the hypersurface Σ being spacelike, timelike or null, respectively. The corresponding Cauchy problems require different types of treatments. By considering the vanishing of the obstruction tensor $\mathcal{O}[g] = \Delta^{\frac{n}{2}-2}\,(\Delta \mathrm{P} + \mathrm{Hess}^g\, J) + \textit{lower order terms}$ instead of the conformal Einstein equation $\mathrm{Ric}[\sigma^2 g] \propto g$ many results have been gained. For example Anderson generalised an existence and stability result by H. Friedrich [Fri86c, Fri86a] to higher even dimensions for asymptotically de Sitter spaces [And05a]. Those are globally hyperbolic, conformally compact Einstein manifolds with positive scalar curvature and spacelike conformal boundary. In this case the boundary has the property that it contains the "end points" of causal geodesics and thus is called conformal future or past. The problem of prescribing sufficient initial data for the equations provided by a vanishing obstruction tensor is solved by considering the coefficients of the Fefferman-Graham expansion. A similar method is then used by Anderson and Chruściel [AC05] to get an existence result for globally hyperbolic, conformally Ricci flat metrics in even dimension. The latter paper uses initial data on a Cauchy hypersurface that intersects the conformal null infinity. A side effect of the geodesic compactification used in [And05a] is a loss of regularity at the boundary. This has been pointed out by D.W. Helliwell [Hel08], who used an almost geodesic compactification to avoid this problem resulting in an improved regularity result.

Provided that the metric is conformally Ricci-flat away from the conformal boundary, Σ is a null hypersurface. In general relativity such manifolds correspond to asymptotically Ricci-flat spacetimes and span a rich field of research. H. Friedrich proposed another solution to the problem of singular behaviour of the equation $\mathrm{Ric}[\sigma^{-2} g] \propto g$ at the singularity set Σ [Fri81a]. He reduced the equations that provide a conformally Ricci-flat Einstein metric to a first-order quasilinear system by introducing the Schouten tensor, a conformally rescaled Weyl tensor, the conformal factor σ and its derivatives $d\sigma$ and $\Delta^g \sigma$ as new unknowns (see [DN98] for a comprehensive review). The emerging system will be referred to as reduced conformal field equations. Friedrich showed symmetrisability of the system and provided extensive existence, uniqueness and stability results [Fri81b, Fri81a, Fri83, Fri86c]. The treatment of the characteristic Cauchy problem with data on a null hypersurface then was brought forward, when A.D. Rendall [Ren90] introduced a method that often can reduce the characteristic problem to an ordinary Cauchy problem. Rendall showed the well posedness of vacuum Einstein field equations if the data are given on two transversally intersecting null hypersurfaces. Rendall's approach and the reduced conformal field equation were later used by J. Kannar [Kan96] to treat the problem of two such null hypersurfaces with one being the conformal boundary. Part of the construction in [Ren90] is the existence of *standard coordinates* in a neighbourhood of points in the intersection of the two null hypersurfaces that are adapted to the problem. The coordinates are a set of spacelike harmonic coordinates parametrising the intersection and two more harmonic null coordinates affinely parametrising the hypersurfaces.

Null cones are a special type of hypersurfaces, which appear as locus of the Cauchy data. They are special in the sense that they obviously are not hypersurfaces at the vertex. The problem with initial data at conformal infinity has for example been treated by Friedrich in dimension 4 by reformulating it in terms of a five dimensional submanifold $\tilde{M}$ of the spin frame bundle $S(M)$, which naturally projects to M [Fri86b]. Aside the asymptotic structure of spacetimes remarkable existence and uniqueness results on the Cauchy problem with data given on a characteristic cone were found by Y. Choquet-Bruhat, P. Chruściel and J.M. Martín-García [CBCMG11b, CBCMG11a]. The conformal wave equations, which were introduced by T.-T. Paetz [Pae13] and then further investigated by Chruściel, Friedrich and Paetz [CP13, Fri13], are then a recent development of the reduced conformal field equations. The conformal wave equations are a method to obtain a non-singular system of partial differential equations with initial data at conformal infinity on a characteristic cone. Up to now no generalisations of the results in dimension $n > 4$ are known to the author.

The methods that have been introduced so far for treating the conformal Cauchy problem can be characterised by the following observations. First there are the results that are at least partially based on the Fefferman-Graham expansion. An important ingredient of that treatment is the assumption of a foliation in a neighbourhood of the conformal boundary such that the conformal boundary appears as a leaf of the foliation. From a different point of view the Fefferman-Graham expansion makes use of coordinates in which the defining function σ is fully prescribed by one coordinate in a neighbourhood of the conformal boundary. Hence only the metric and if necessary its derivatives are considered to be unknowns of a system of PDEs. By using the Fefferman-Graham expansion, one is restricted to non-degenerate initial data at the conformal boundary and the author is not aware of a way that can be used to apply the Fefferman-Graham expansion to degenerate initial data, as it appears in Lorentzian signature for almost scalar flat almost Einstein structures with data at the singularity set Σ. An expansion-method nevertheless has been introduced by H. Friedrich in dimension 4 [Fri13] for just the latter Cauchy problem with data at a null cone. However, the method does not easily generalise to higher dimensions due to the usage of the Newman-Penrose formalism. On the other hand the characteristic initial data problem can be approached by a second type of treatment, which considers the metric, the conformal factor and its derivatives as unknowns to new systems of PDEs, namely the conformal field equations and the conformal wave equations. We will focus on the last type of treatment. More precisely we on the one hand will introduce an approach to generalise the conformal wave equations to higher even dimensions and on the other hand we will construct local coordinates in which the conformal factor and null pregeodesics originating at the vertex of the characteristic cone are prescribed. We believe that this may lead to a new treatment of the characteristic Cauchy problem with data at a null cone at the conformal boundary. The results and content of this thesis are as follows.

We will review basic methods to get reduced PDEs for almost Einstein structures, with the metric tensor as an unknown, in an index-free notation. This includes the wave-map gauge, the reduced Ricci tensor, the reduced Laplace operator and the conformal wave equations. By replacing the Bach tensor in [Pae13] with the obstruction tensor we propose a method to get conformal wave equations in higher even dimensions. Finally we will construct coordinates that are adapted to the null cone at conformal infinity. In contrast to the latter results, where the null direction along the cone is parametrised by affine coordinates we will drop this requirement in order to obtain coordinates that still parametrise the null direction (but not affinely) at the null cone and in addition fully prescribe the boundary defining function σ in a neighbourhood U of a vertex. In particular we provide coordinates $x : M \supset U \to \mathbb{R}^n$ such that up to a sign $\sigma = \pm\left(-\left(x^0\right)^2 + \left(x^1\right)^2 + \cdots + \left(x^{n-1}\right)^2\right)$ and in addition null geodesics originating at the vertex are mapped to the line $\mathbb{R}(1, \mathbf{e})$ with unit vector $\mathbf{e} \in \mathbb{R}^{n-1}$ (Theorem 5.3.38). We will then show that the metric of the corresponding almost Einstein structure has a simple form in further advanced coordinates along Σ that are based on our construction (Equation (5.29)).

En passant we will show that any diffeomorphism $f : S^n \to S^n$ on the sphere, which is sufficiently close to the identity in uniform norm, can be lifted to a global section g of the trivial $\underline{\mathfrak{so}}(n+1)$ bundle over S^n that fulfils $f(x) = \exp(g(x)) \cdot x$ (Proposition 5.3.21). This eventually gives a characterisation of diffeomorphisms on the sphere by whether they lift to a section G of the trivial $SO(n+1)$ bundle over S^n with $f(x) = G(x) \cdot x$ or whether they do not. As a consequence different connected components in the space of diffeomorphisms on the sphere can thus be distinguished in uniform topology by this property (Corollary 5.3.22). Although these results are detached from the subject of the remaining thesis, they provide important tools for the proofs therein.

ORGANISATION OF THE THESIS

The thesis is organised as follows. The first chapter introduces basic definitions and statements of pseudo-Riemannian and conformal geometry. Also a survey of results on almost Einstein structures will be given. The aim is to present the content in a mostly self-evident way, such that almost no further literature is necessary to understand the notation used in the thesis.

Chapter 2 provides known examples for conformal completions of Lorentzian Einstein manifolds and interprets the results as almost Einstein structures.

Results in the mathematical and physical neighbourhood of the thesis are presented in more detail in chapters 3 and 4. Since some of the statements in the literature are calculated with the use of some sort of coordinate, abstract or frame indices, a first intention is to reproduce the results in a notation without such indices. The method of conformal wave equations used for the treatment of the characteristic initial data problem on conformally Ricci-flat Einstein manifolds is provided in chapter 4. It is developed in such a way that it contains an ansatz for a generalisation to higher even dimensions.

The main results of the thesis are finally presented in chapter 5. The chapter starts with an analysis of the characteristics of the singularity set of Lorentzian almost Einstein structures (M, g, σ) that are almost scalar flat. A main part of the chapter is the construction of special coordinates in a neighbourhood of certain vertices of the singularity set that are adapted to the topology and causality of the singularity set and implicitly determine the boundary defining function σ. The chapter closes with applying the coordinates to the calculation of the metric along the singularity set.

The thesis ends with an outlook on future fields of investigation that have been tangent to the thesis but have not been considered in more detail.

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1 DIFFERENTIAL GEOMETRY

This chapter will provide an introduction to the fundamental definitions and concepts that are the basis of this thesis. The intention is to present it in a mostly self-evident way. The chapter starts with a section on tensors, connections and operations with it. This thesis deals with pseudo-Riemannian manifolds in Lorentzian signature $(- + \cdots +)$. The causal structure of such manifolds is the topic of a separate section, which is followed by a section on special issues of the matrix Lie group $SO(n)$. The focus will be on $SO(n)$'s property of being a manifold that admits an exponential map for each of its points. The chapter concludes with a section on conformally Einstein manifolds. There are different concepts of introducing the term Einstein metric or Einstein manifold that depend on the environment where it is used. Their common ground will be part of that section.

1.1 PSEUDO-RIEMANNIAN GEOMETRY

This section will be a survey of fundamental concepts, definitions and notations in pseudo-Riemannian geometry. The objective is to reduce the need of research in secondary literature to a minimum. After defining tensors and derivatives, the section will introduce important differential and curvature operators and their properties. The section ends with a short survey of vector fields and flows. Partially proofs will be provided if they are essential to the understanding of the thesis. In particular this is done if it introduces an uncommon method, which is used later on.

At the basis of the mathematical framework is a pseudo-Riemannian manifold (M, g) of dimension n. We will denote by ∇^g the *Levi-Civita connection* with respect to the metric g. If not misleading, we will drop the g and simply write ∇.

Tensor Bundles and Derivatives

The notation for the fundamental tensor bundles is

TM	for the tangent bundle on M. Its elements will be called vectors or $(0, 1)$-tensors.
T^*M	for the cotangent bundle on M. Its elements will be called covectors or $(1, 0)$-tensors.
$T^{p,q}M := \bigotimes^p T^*M \otimes \bigotimes^q TM$	for the tensor bundle of tensors of valence (p, q). Its elements will be called (p, q)-tensors.
$\mathcal{T}^{p,q}M := \Gamma\left(T^{p,q}M\right)$	for the sections of the tensor bundle $T^{p,q}M$. Its elements will be called tensor fields. If in the contexts there is no confusion, they will be called just tensors. The sections of TM will be denoted $\mathcal{T}M$
$\Omega^p(M) := \Gamma\left(\bigwedge^p T^*M\right)$	for the p-forms on M. $\Omega^1(M)$ will also be denoted $\Omega(M)$.
$\mathfrak{X}(M) := \Gamma(TM)$	for vector fields on M. As a short form vector fields will also be referred to as vectors.
$TN^\perp$	for the normal bundle of a pseudo-Riemannian submanifold $N \subset M$ in M.
$\mathfrak{X}(N)^\perp$	for the sections of the normal bundle.

$T^{p,q}_\gamma M := \gamma^* T^{p,q} M$	for the pullback bundle of $T^{p,q}M$ by a curve $\gamma : I \to M$. It is a vector bundle over I with fibre $T^{p,q}_{\gamma(t)}M$.
$\mathcal{T}^{p,q}_\gamma M := \Gamma\left(T^{p,q}_\gamma M\right)$	for the tensor fields along a curve γ.
$\mathfrak{X}_\gamma(M) := \Gamma\left(T_\gamma M\right)$	for vector fields along a curve γ.

The notation for fundamental derivatives is

$\mathcal{L}_X : \mathcal{T}^{p,q}M \to \mathcal{T}^{p,q}M$	for the *Lie derivative*.
$D : \mathcal{T}M \to \mathcal{T}^{1,1}M$	for an arbitrary connection on $\mathcal{T}M$.
$\nabla^g : \mathcal{T}M \to \mathcal{T}^{1,1}M$	for the Levi-Civita connection. It will also be denoted ∇ if the underlying metric is fixed.
$D_{\dot\gamma} T \in \mathcal{T}^{p,q}_\gamma M$	for the *covariant derivative* $\frac{D\gamma}{dt}T(t) = (DT)_{\gamma(t)}\left(\dot\gamma(t), \dots\right)$ of a tensor field $T \in \mathcal{T}^{p,q}M$ along a curve $\gamma : I \to M$.
$\ddot\gamma \in \mathcal{T}_\gamma M$	for the covariant derivative $D_{\dot\gamma}\dot\gamma$ of $\dot\gamma$ along the curve γ. A curve with $\ddot\gamma \equiv 0$ is called D-geodesic.
$\mathcal{P}^D_\gamma : T^{p,q}_{\gamma(a)}M \to T^{p,q}_{\gamma(b)}M$	for the *parallel translation* of a tensor $T \in T^{p,q}_{\gamma(a)}M$ along $\gamma : [a,b] \to M$ with respect to D.
$T^D \in \mathcal{T}^{2,1}M$	for the *torsion* tensor of a connection D. A connection with vanishing torsion tensor is called torsion-free.

A connection admits a generic extension to a connection on tensor fields of arbitrary valence by requiring $D_X f := X(f)$ on smooth functions and demanding a Leibniz rule for tensor products. It will be denoted with the same symbol $D : \mathcal{T}^{p,q}M \to \mathcal{T}^{p+1,q}M$ and is defined by

$$(DT)(X, \theta_1, \dots, \theta_{p+q}) := X\left(T(\theta_1, \dots, \theta_{p+q})\right) - \sum_{i=1}^{p+q} T\left(\dots, D_X\theta_i, \dots\right) \tag{1.1}$$

where X is a vector field, while θ_i are either $(1,0)$- or $(0,1)$-tensor fields. A frequently-used notation is $D_X T := (DT)(X, \dots)$.

Definition A vector field $X \in \mathfrak{X}(M)$ on (M, g) is said to be *p-synchronous* with respect to D at $p \in M$ if

$$(DX)_p = 0.$$

Any vector $X \in T_pM$ can locally be extended to a p-synchronous vector field via parallel transport along radial geodesics originating in p.

Contractions, Traces and Dualisation

The next paragraphs will introduce a notation for contractions, metric traces and metric dualisation on a semi-Riemannian manifold. The basic notations are

$\omega^\sharp \in \mathfrak{X}(M)$	for the metric dual of a 1-form $\omega \in \Omega(M)$ defined by $g(\omega^\sharp, Y) := \omega(Y)$.
$X^\flat \in \Omega(M)$	for the metric dual of a vector field $X \in \mathfrak{X}(M)$ defined by $X^\flat(Y) := g(X, Y)$.

$T^{\sharp_i} \in \mathcal{T}^{p-1,q+1}M$, $T^{\flat_j} \in \mathcal{T}^{p+1,q-1}M$	for the metric dualisation of a tensor $T \in \mathcal{T}^{p,q}M$ in its i's or j's argument with $i \leq p$ and $j \leq q$. This is defined by $T^{\sharp_i}(\ldots, X_{i-1}, X_i^\flat, X_{i+1}, \ldots) := T(\ldots, X_{i-1}, X_i, X_{i+1}, \ldots)$ or $T^{\flat_i}(\ldots, \omega_{j-1}, \omega_j^\sharp, \omega_{i+1}, \ldots) := T(\ldots, \omega_{i-1}, \omega_i, \omega_{i+1}, \ldots)$. If the index is dropped, $T^\sharp$ and $T^\flat$ will refer to dualisation in the first component.
$\operatorname{tr}^g_{j,k} W$	for the metric trace in two covariant or contravariant arguments. A tensor of valence (p,q) is either mapped to a tensor of valence $(p-2,q)$ or $(p,q-2)$. In terms of an orthonormal frame $\{e_i\}$ with $\epsilon_i = g(e_i, e_i)$ the trace of the tensor W may be expressed as $\operatorname{tr}^g_{j,k} W = \sum_{i=1}^n \epsilon_i W(\ldots, e_i, \ldots, e_i, \ldots)$, such that the e_i are arguments at j-th and k-th position.
$\mathcal{C}_{j,k} : \mathcal{T}^{p,q}M \to \mathcal{T}^{p-1,q-1}M$	for the contraction or natural pairing of a tensor in one contravariant and one covariant argument.
$P_A : \mathcal{T}^{p,0}M) \to \Omega^p(M)$	for the *antisymmetrisation* of a tensor. It is given by $(P_A T)(X_1, \ldots, X_p) := \frac{1}{p!} \sum_{\sigma \in S_p} (\operatorname{sgn} \sigma) T\left(X_{\sigma(1)}, \cdots, X_{\sigma(p)}\right)$.

Some short forms of the above definitions will be used frequently. If the trace or the contraction of a tensor of valence $(2,0)$, $(1,1)$ or $(0,2)$ is considered the indices will be neglected. For example $\tau^g = \operatorname{tr}^g \operatorname{Ric}^g$ instead of $\tau^g = \operatorname{tr}^g_{1,2} \operatorname{Ric}^g$ or $n = \mathcal{C}\,\mathrm{id}$ instead of $n = \mathcal{C}_{1,2}\,\mathrm{id}$. Sometimes it is useful to introduce a short notation for the trace-free part of a symmetric tensor S. This will be

$$S_0 := S - \frac{\operatorname{tr}^g S}{n} g.$$

The metric independent pairing $\mathcal{C}_{ij}$ can be used to rewrite metric traces or metric dual. The trace of a $(0,2)$-tensor O for example is the double contraction $\mathcal{C}_{1,2}(\mathcal{C}_{1,3} g \otimes O)$ while the dual of a vector X may also be written as $X^\flat = \mathcal{C}_{1,3} g \otimes X$. Using the metric on TM and its dual on T^*M any dualisation or trace can be written as a contraction in a similar way. Frequently-used metric duals are those of the metric tensor itself and of the Hessian of $f \in C^\infty(M)$. There are different equivalent notations for the $(2,0)$-*Hessian* of of a smooth map $f \in C^\infty(M)$ on (M,g). That are

$$\operatorname{Hess}^g f := \nabla^g \nabla^g f \tag{1.2}$$
$$\operatorname{Hess}^g f(X,Y) = g\left(\nabla^g_X \operatorname{grad} f, Y\right) \tag{1.3}$$
$$= X(Y(f)) - df(\nabla^g_X Y). \tag{1.4}$$

Then the mentioned dualisations are

$$(\operatorname{Hess}^g f)^\sharp = \nabla^g \operatorname{grad} f \quad \text{and} \tag{1.5}$$
$$g^\sharp = \mathrm{id}, \tag{1.6}$$

where in the second line the $(1,1)$-tensor $g^\sharp$ may be used in its interpretation as identity morphism on $\mathfrak{X}(M)$ or $\Omega(M)$.

Differential Operators

Let $D : \mathcal{T}^{p,q}M \to \mathcal{T}^{p+1,q}M$ be a torsion-free connection on (M,g). It admits a canonical formal adjoint with respect to the metric

$$D^* : \mathcal{T}^{(p+1,q)}M \to \mathcal{T}^{p,q}M. \tag{1.7}$$

Taking a orthonormal frame $\{e_i\}$, this can locally be expressed by [Bes08, 1.55]

$$(D^* T)(\theta_1, \ldots, \theta_{p+q}) = -\sum_{i \in \{1,\ldots,n\}} \epsilon_i (D_{e_i} T)(e_i, \theta_1, \ldots, \theta_{p+q}), \tag{1.8}$$

with $T \in \mathcal{T}^{p+1,q}$ and either $\theta_i \in \mathfrak{X}(M)$ or $\theta_i \in \Omega(M)$. By identification of a $(p+1,q)$-tensor with a map $\omega : \mathcal{T}^{(p+1,0)} \to \mathcal{T}^{(0,q)}$ this may also be written as $(D^*\omega)(X_1,\dots,X_p) := -\sum_{i\in\{1,\dots,n\}} \epsilon_i \,(D_{e_i}\omega)(e_i, X_1,\dots,X_p)$ for vector fields $X_i \in \mathfrak{X}(M)$. Using the metric trace, the formal adjoint can be written as $D^*T = -\operatorname{tr}^g_{1,2} DT$. Its generalisation to arbitrary traces $-\operatorname{tr}^g_{1,j} DT$ with $j \in \{2,\dots,p+q+1\}$ will be called *divergence* and is denoted by

$$\operatorname{div}^D_j \omega(X_1,\dots,X_p) := -\sum_{i\in\{1,\dots,n\}} \epsilon_i \,(D_{e_i}\omega)(\dots,X_{j-1},e_i,X_j,\dots). \tag{1.9}$$

If the connection is the Levi-Civita connection, the notation will be div_j. If the index is suppressed, it will always refer to the $\operatorname{tr}^g_{1,2}$-trace. If the divergence is applied to a symmetric tensor field S, $\operatorname{div}^D_j S = \operatorname{div}^D_k S$ holds for any j and k. The index will therefore be dropped in case of symmetric tensors.

The *exterior derivative* on forms $d : \Omega^p(M) \to \Omega^{p+1}(M)$ and the *codifferential* $\delta : \Omega^{p+1}(M) \to \Omega^p(M)$ are differential operators that can be expressed in terms of the Levi-Civita connection on (M,g). Let $\omega \in \Omega^p(M)$ be a p-form on M, then

$$d\omega(X_0,\dots,X_p) = \sum_{i\in\{0,\dots,p\}} (-1)^i \left(\nabla^g_{X_i}\omega\right)(X_0,\dots,\hat{X}_i,\dots,X_p) \tag{1.10}$$

$$\delta\omega(X_1,\dots,X_{p-1}) = -\sum_{i\in\{1,\dots,p-1\}} \epsilon_i \left(\nabla^g_{e_i}\omega\right)(e_i,X_1,\dots,X_{p-1}) \tag{1.11}$$

where the hat denotes skipping of that element. Using the metric trace and the antisymmetrisation the notation can be shorten

$$d\omega = (p+1)P_A(\nabla^g\omega). \tag{1.12}$$

$$\delta\omega = -\operatorname{tr}^g_{1,2} \nabla^g\omega. \tag{1.13}$$

Important second-order, linear differential operators on (M,g) are the different Laplacians.

$\Delta_p : \Omega^p(M) \to \Omega^p(M)$	is the *Hodge Laplace operator* or *Laplace-de Rham operator* on (M,g). It maps a p-form as $\omega \mapsto (d\delta + \delta d)\omega$.
$\Delta^D : \Gamma(E) \to \Gamma(E)$	is the *Bochner Laplacian* or *rough Laplacian* on a vector bundle (E,π,M) with connection $D : \Gamma(E) \to \Gamma(T^*M \otimes E)$. The formal adjoint $D^* : \Gamma(T^*M\otimes E) \to \Gamma(E)$ is given by $D^*\tilde{T} = -\operatorname{tr}^g_{1,2}(D^1\tilde{T})$, where $D^1 = \nabla^g \otimes \mathrm{id}_E + \mathrm{id}_{\Omega(M)} \otimes D$ [Bes08, section 1.I]. The Laplacian then is defined by $\Delta^D T = D^*DT$. If applied to the trivial line bundle over M, it is also called Laplace-Beltrami operator.

Of special interest here is the Bochner Laplacian arising from the Levi-Civita connection on $E = \Omega^p(M) \subset \mathcal{T}^{p,0}M$. It is related to the Hodge Laplacian via a Weitzenböck identity. For $p = 1$, the identity is as follows.

Lemma 1.1.1. *Let (M,g) be a pseudo-Riemannian manifold, ∇ the Levi-Civita connection and $\omega \in \Omega^1(M)$ a 1-form on (M,g). Then the following Weitzenböck identity connects the Hodge Laplacian Δ_1 with the Bochner Laplacian Δ^∇.*

$$\Delta_1\omega = \Delta^\nabla\omega + \operatorname{Ric}^\sharp(\omega) \tag{1.14}$$

The Ricci tensor Ric will be defined on page 13. A proof using methods that are exploited throughout the thesis is given in the appendix. One has the following consequence for the commutator of the Bochner Laplacian and the Levi-Civita connection.

Corollary 1.1.2. *Let $f \in C^\infty(M)$ be a smooth function on M. Then it holds*

$$[\Delta^\nabla, \nabla] f = -\operatorname{Ric}^\sharp(\operatorname{grad} f). \tag{1.15}$$

The equation can be calculated, if one uses $dh = \nabla h$ on maps and $\delta d f = \Delta^\nabla f$. Then Equation (1.15) is the result of

$$\begin{aligned} \nabla \Delta^\nabla f &= (d\delta + \delta d) df \\ &\overset{(1.14)}{=} \Delta^\nabla \nabla f + \operatorname{Ric}^\sharp(df). \end{aligned}$$

Curvature Tensors

The next section will fix the notation on curvature tensors that are deduced from a connection and in particular from the Levi-Civita connection ∇ on (M, g). The generic *curvature tensor* of a connection D on $\mathcal{T}M$ is denoted

$$\mathrm{R}^D(X, Y) = D_X D_Y - D_Y D_X - D_{[X,Y]}.$$

On the other hand the curvature tensor of the Levi-Civita connection ∇ on (M, g) will also be denoted $\mathrm{R}^\nabla = \mathrm{R}^g$. For the $(4, 0)$-Riemann curvature tensor, the following convention is used

$$\mathrm{R}^g(X, Y, V, W) = g(V, \mathrm{R}^g(X, Y)W). \tag{1.16}$$

Depending on the order of the components this definition may lead to a sign in comparison with other conventions. Derivatives of the metric are hidden if one uses $\mathrm{R}^g(X, Y, U, V) = g\left(U, \nabla_X \nabla_Y V - \nabla_Y \nabla_X V - \nabla_{[X,Y]} V\right)$ for the Riemann tensor. For calculations it will be important to have the Riemann tensor written in terms of Lie-derivatives and the metric instead of the Levi-Civita connection. The benefit is to make derivatives of the metric tensor explicit.

Remark 1.1.3. The Riemann tensor $R(X, Y, V, W)$ can be expressed completely in terms of Lie derivatives of g, X, Y, V and W. For that one first observes

$$\begin{aligned} (\mathcal{L}_X \mathcal{L}_Y g)(V, W) = {} & g(\nabla_X \nabla_V Y, W) + g(\nabla_X \nabla_W Y, V) + g(\nabla_V Y, \nabla_X W) \\ & - (\mathcal{L}_Y g)(W, [X, V]) - (\mathcal{L}_Y g)(V, [X, W]) + g(\nabla_X V, \nabla_W Y). \end{aligned}$$

Commuting covariant derivatives as $\nabla_X Y = [X, Y] + \nabla_Y X$ and expressing derivatives of type $Z(g(X, Y))$ by $(\mathcal{L}_Z g)(X, Y) + g([Z, X], Y) + g(X, [Z, Y])$ then gives

$$\begin{aligned} 2g(\mathrm{R}(X, Y)V, W) = {} & (\mathcal{L}_X \mathcal{L}_V g)(Y, W) - (\mathcal{L}_X \mathcal{L}_W g)(Y, V) + (\mathcal{L}_Y \mathcal{L}_W g)(X, V) - (\mathcal{L}_Y \mathcal{L}_V g)(X, W) \\ & + 2\left(g(\nabla_X V, \nabla_Y W) - g(\nabla_X W, \nabla_Y V)\right) + g([X, Y], [V, W]) \\ & + (\mathcal{L}_V g)(W, [X, Y]) + (\mathcal{L}_V g)(Y, [X, W]) + (\mathcal{L}_V g)(X, [W, Y]) \\ & - (\mathcal{L}_W g)(V, [X, Y]) - (\mathcal{L}_W g)(Y, [X, V]) - (\mathcal{L}_W g)(X, [V, Y]) \\ & - (\mathcal{L}_X g)(Y, [V, W]) + (\mathcal{L}_Y g)(X, [V, W]). \end{aligned}$$

To get rid of the remaining terms that involve the Levi-Civita connection, one can locally use the equality $g(\nabla_X Y, \nabla_V W) = \sum_i \epsilon_i g(\nabla_X Y, e_i) g(e_i, \nabla_V W)$. Each of the factors on the right-hand side then can be rewritten in terms of Lie derivatives if the Koszul formula is used

$$\begin{aligned} 2g(\nabla_X Y, Z) = {} & (\mathcal{L}_X g)(Y, Z) + (\mathcal{L}_Y g)(X, Z) - (\mathcal{L}_Z g)(X, Y) \\ & + g([X, Y], Z) + g([Y, Z], X) - g([Z, X], Y). \end{aligned}$$

A combination of the last two formulas then gives the claim.

The curvature tensor may be applied to arbitrary tensor fields. It then is beneficial to have another notation. Let D be a connection on $\Gamma(TM)$ and let the same symbol stands for the connection on $\mathcal{T}^{p,q}M$, defined by Equation (1.1). Then one has the following lemma.

Lemma 1.1.4. *Let D be a torsion-free connection. The $(p+q+2, p+q)$-curvature tensor R^D applied to a tensor field $T \in \Gamma(T^{p,q}M)$ can then be written as*

$$\left(R^D(X,Y)T\right)(\cdot,\dots,\cdot) = (DDT)(X,Y,\cdot,\cdots,\cdot) - (DDT)(Y,X,\cdot,\cdots,\cdot). \tag{1.17}$$

Let $\{\theta_i\}_{i=1,..,s}$ be a set of vector fields or covector fields, where $s = p+q$. Then a calculation gives

$$\begin{aligned}(DDT)(X,Y,\theta_1,\dots,\theta_s) &= D_X\left((D_Y T)(\theta_1,\dots,\theta_s)\right) - (DT)(D_X Y,\theta_1,\dots,\theta_s) \\ &\quad - \sum_{i\in\{1,\dots,s\}} (DT)(Y,\dots,D_X\theta_i,\dots) \\ &= (D_X D_Y T)(\theta_1,\dots,\theta_s) + \sum_{i=1}^{s} (D_Y T)(\dots,D_X\theta_i,\dots) \\ &\quad - \left(D_{D_X Y}T\right)(\theta_1,\dots,\theta_s) - \sum_{i=1}^{s}(D_Y T)(\dots,D_X\theta_i,\dots)\end{aligned}$$

The sums cancel each other. Antisymmetrisation in X and Y then gives Equation (1.17). The curvature tensor of the connection D if acting on $\mathcal{T}^{p,q}M$ can be related to the curvature tensor of D if acting on vector fields and covector fields.

Lemma 1.1.5. *Let $T \in \mathcal{T}^{p,q}M$ be a tensor field, $X, Y \in \mathfrak{X}(M)$ and θ_i vector or covector fields. Then it holds*

$$(R^D(X,Y)T)(\theta_1,\dots,\theta_s) = -\sum_{i=1}^{s} T(\dots,R^D(X,Y)\theta_i,\dots), \tag{1.18}$$

where $s = p+q$.

The equation is calculated as follows

$$\begin{aligned}(D_X D_Y T)(\theta_1,\dots,\theta_p) &= D_X\left((D_Y T)(\theta_1,\dots,\theta_s)\right) - \sum_{i=1}^{s}(D_Y T)(\dots,D_X\theta_i,\dots) \\ &= X\left(Y\left(T(\theta_1,\dots,\theta_s)\right)\right) \\ &\quad - D_X\left(\sum_{i=1}^{s} T(\dots,D_Y\theta_i,\dots)\right) - D_Y\left(\sum_{i=1}^{s} T(\dots,D_X\theta_i,\dots)\right) \\ &\quad + \sum_{i\neq j} T(\dots,D_Y\theta_j,\dots,D_X\theta_i,\dots) + \sum_{i=1}^{s} T(\dots,D_Y D_X\theta_i,\dots) \\ &= Y\left(X\left(T(\theta_1,\dots,\theta_s)\right)\right) + [X,Y]\left(T(\theta_1,\dots,\theta_s)\right) \\ &\quad - D_X\left(\sum_{i=1}^{s} T(\dots,D_Y\theta_i,\dots)\right) - D_Y\left(\sum_{i=1}^{s} T(\dots,D_X\theta_i,\dots)\right) \\ &\quad + \sum_{i\neq j} T(\dots,D_Y\theta_j,\dots,D_X\theta_i) \\ &\quad + \sum_{i=1}^{s} T(\dots,\left(R^D(Y,X) + D_X D_Y + D_{[Y,X]}\right)\theta_i,\dots) \\ &= (D_Y D_X T)(\theta_1,\dots,\theta_s) + \left(D_{[X,Y]}T\right)(\theta_1,\dots,\theta_s) \\ &\quad - \sum_{i=1}^{s} T(\dots,R^D(X,Y)\theta_i,\dots)\end{aligned}$$

The claim follows directly. In particular for a form $\omega \in \Omega(M)$ one has $\left(\mathrm{R}^D(X,Y)\omega\right)(Z) = -\omega\left(\mathrm{R}^D(X,Y)Z\right)$. By making the notation R^D_{vf} for the $(3,1)$-curvature tensor on vector fields, the curvature may be written in terms of R^D_{vf}. Let $Z_i \in \mathfrak{X}(M)$ be vector fields and $\omega_i \in \Omega(M)$ be forms then the *generalised Ricci identity* reads as

$$\begin{aligned}&(R^D(X,Y)T)(Z_i,\dots,Z_p,\omega_1,\dots,\omega_q) = \\ &\qquad -\sum_{i\in\{1,\dots,p\}} T(\dots,R^D_{vf}(X,Y)(Z_i),\dots) + \sum_{i\in\{1,\dots,q\}} T(\dots,R^D_{vf}(X,Y)(\omega_i),\dots),\end{aligned} \tag{1.19}$$

where $R^D_{vf}(X,Y)(\omega) := \omega\left(\mathrm{R}^D_{vf}(X,Y)\cdot\right)$ and $R^D_{vf}(X,Y)(Z) = \mathrm{R}^D_{vf}(X,Y)Z$. The last notation motivates the definition of a generalised product of $(1,1)$-tensors with arbitrary tensors as follows.

Definition Let $S \in \mathcal{T}^{1,1}M$ and $T \in \mathcal{T}^{p,q}M$ be two tensor fields. By writing $S(X) := S(X,\cdot)$ for vectors X and $S(\omega) := S(\cdot,\omega)$ for covectors, the *Ricci product* $S \cdot T \in \mathcal{T}^{p,q}M$ of S and T is defined as:

$$\begin{aligned}(S\cdot T)\left(Z_1,\dots,Z_p,\omega_1,\dots,\omega_q\right) := & -\sum_{i=1}^{p} T\left(Z_1,\dots,S(Z_i),\dots,Z_p,\omega_1,\dots,\omega_q\right) \\ & +\sum_{i=1}^{q} T\left(Z_1,\dots,Z_p,\omega_1,\dots,S\left(\omega_i\right),\dots,\omega_q\right).\end{aligned} \tag{1.20}$$

On smooth maps f the Ricci product is defined to give the trivial map

$$S\cdot f := 0. \tag{1.21}$$

In particular this implies $S\cdot X = S(X)$ for vector fields and $S\cdot\omega = -S(\omega)$ for forms. On the other hand the generalised Ricci identity now simply reads $R^D(X,Y)T = R^D_{vf}(X,Y)\cdot T$. As an application of the Ricci product, the curvature tensor of the connection D on $T \in \mathcal{T}^{p,q}M$ can be written as $\mathrm{R}^D(X,Y)T = \mathrm{R}^D_{vf}(X,Y)\cdot T$, which motivates a further definition.

Definition 1.1.6. Let T be a tensor field of valence (p,q) and R^D the curvature tensor of a torsion-free connection D on TM, extended by Equation (1.1). Then the following notation is used

$$(\mathrm{R}^D_{vf}\cdot T)(X,Y,\dots) := \left(\mathrm{R}^D(X,Y)T\right)(\dots). \tag{1.22}$$

Corollary 1.1.7. *Consider T to be an arbitrary tensor field, S to be a $(1,1)$-tensor field and D, $\tilde{D}$ be two torsion-free connections on the tangent bundle of M. Let R^D be the curvature tensor of D. Then the Ricci product and $\mathrm{R}^D_{vf}\cdot T$ fulfil the following Leibniz rules*

$$\tilde{D}_Z\left(S\cdot T\right) = (\tilde{D}_Z S)\cdot T + S\cdot\left(\tilde{D}_Z T\right) \tag{1.23}$$

$$\begin{aligned}\left(\tilde{D}\left(\mathrm{R}^D_{vf}\cdot T\right)\right)(Z,X,Y,\dots) &= (\tilde{D}\,\mathrm{R}^D_{vf})(Z,X,Y)\cdot T + \mathrm{R}^D_{vf}(X,Y)\cdot\left(\tilde{D}_Z T\right) \\ &\overset{(1.22)}{=} (\tilde{D}\,\mathrm{R}^D_{vf})(Z,X,Y)\cdot T + \mathrm{R}^D(X,Y)\left(\tilde{D}_Z T\right).\end{aligned} \tag{1.24}$$

Since the Ricci product is a short notation for a sum of contracted tensor products, the Leibniz rule is a consequence of the Leibniz rule on tensor products. Now let ∇ be the Levi-Civita connection of (M,g) and R^g be its curvature tensor. The *Ricci tensor* is defined as

$$\mathrm{Ric}^g(X,Y) := \sum_{i\in\{1,\dots,n\}} \epsilon_i g(R^g(X,e_i)e_i,Y), \tag{1.25}$$

where $\{e_i\}$ is a local orthonormal frame. Metric dualisation in one argument can be written as $\mathrm{Ric}^\sharp(X) = \sum_{i\in\{1,\dots,n\}} \epsilon_i R^g(X,e_i)e_i$. Its trace will be written

$$\tau^g := \mathrm{tr}^g\,\mathrm{Ric}^g. \tag{1.26}$$

If Ricci curvature or scalar curvature are understood as maps on the space of metrics on M, they may also be denoted $\mathrm{Ric}[g]$ and $\tau[g]$.

Bianchi Identities

Important for further calculations are the Bianchi identities and their contractions, which will be recalled now. The first and second Bianchi identities are

$$0 = \mathrm{R}^g(X,Y)Z + \mathrm{R}^g(Y,Z)X + \mathrm{R}^g(Z,X)Y \tag{1.27}$$

$$0 = \left(\nabla_X \mathrm{R}^g\right)(Y,Z) + \left(\nabla_Y \mathrm{R}^g\right)(Z,X) + \left(\nabla_Z \mathrm{R}^g\right)(X,Y) \tag{1.28}$$

$$\mathrm{R}^g(X,Y,V,W) = \mathrm{R}^g(V,W,X,Y) = -\mathrm{R}^g(X,Y,W,V) = -\mathrm{R}^g(Y,X,V,W), \tag{1.29}$$

where $X, Y, Z, V, W \in \mathfrak{X}(M)$. A first important consequence concerning the contraction of tensor with symmetries of the Riemann tensor and fulfilling the first Bianchi identity with another tensor is the following.

Corollary 1.1.8. *Let T be a $(4,0)$-tensor with symmetries of the $(4,0)$-Riemann tensor and B be a symmetric $(2,0)$-tensor. Then any double metric trace of the tensor product of those two tensors will give a symmetric $(2,0)$-tensor, i.e.* $\operatorname{tr}^g_{i,j}\left(\operatorname{tr}^g_{k,l} T \otimes B\right)$ *is a symmetric tensor for all $k \neq l \in \{1,\ldots,6\}$ and $i \neq j \in \{1,\ldots,4\}$.*

A proof is provided in the appendix. Contracting the second Bianchi identity leads to the following well-known equation for the divergence of the $(4,0)$-Riemann tensor

$$(\operatorname{div}^g R^g)(X,Y,Z) = \left(\nabla_Z \operatorname{Ric}^g\right)(Y,X) - \left(\nabla_Y \operatorname{Ric}^g\right)(Z,X), \tag{1.30}$$

while contracting it a second time gives another well-known result for the divergence of the Ricci tensor

$$\operatorname{div}^g \operatorname{Ric}^g = -\frac{1}{2} d\tau^g. \tag{1.31}$$

Definition The *Schouten tensor* is defined as

$$\mathrm{P}^g = \frac{1}{n-2}\left(\operatorname{Ric}^g - J^g g\right) \tag{1.32}$$

where $J^g := \operatorname{tr}^g \mathrm{P}^g = \frac{1}{2(n-1)}\tau^g$.

The index g will be omitted in the rest of this section, such that $\nabla = \nabla^g$, $\mathrm{R} = \mathrm{R}^g$, $\operatorname{tr} = \operatorname{tr}^g$ etc.. Using $\nabla \operatorname{Ric} = (n-2)\nabla \mathrm{P} + dJ \otimes g$ the divergence of the Riemann and the Ricci tensor can be rewritten in terms of the Schouten tensor P and its trace J. Equation (1.30) then is equivalent to

$$\begin{aligned} 0 = {} & \operatorname{div} R(X,Y,Z) + (n-2)\left(\left(\nabla_Y \mathrm{P}\right)(X,Z) - \left(\nabla_Z \mathrm{P}\right)(X,Y)\right) \\ & + dJ(Y)g(X,Z) - dJ(Z)g(X,Y) \end{aligned} \tag{1.33}$$

while Equation (1.31) is equivalent to

$$0 = \operatorname{div} \mathrm{P} + dJ. \tag{1.34}$$

The *Kulkarni-Nomizu product* of two symmetric (2,0)-tensors T_1 and T_2 is defined by

$$\begin{aligned} &(T_1 \owedge T_2)(X,Y,Z,V) = \\ &\quad T_1(X,Z)T_2(Y,V) + T_1(Y,V)T_2(X,Z) - T_1(X,V)T_2(Y,Z) - T_1(Y,Z)T_2(X,V). \end{aligned} \tag{1.35}$$

This produxt is symmetric, i.e. $M \owedge N = N \owedge M$ and furthermore has the following properties.

Lemma 1.1.9. *Let M be a symmetric $(2,0)$-tensor, then the Kulkarni-Nomizu product $M \owedge g$ fulfils*

$$\operatorname{tr}_{1,3}(M \owedge g) = \operatorname{tr}(M)g + (n-2)M \tag{1.36}$$

$$\begin{aligned} \left(\operatorname{div} M \owedge g\right)(X,Y,Z) = {} & (\operatorname{div} M)(Y)g(X,Z) - (\operatorname{div} M)(Z)g(X,Y) \\ & + (\nabla_Z M)(X,Y) - (\nabla_Y M)(X,Z). \end{aligned} \tag{1.37}$$

The proof is left to the appendix.

Definition The trace-free part of the $(4,0)$-Riemann tensor is the *Weyl tensor*. It is denoted

$$\mathrm{W} = \mathrm{R} - \mathrm{P} \owedge g \tag{1.38}$$

or W^g if dependence on the metric is needed. In dimension $n \geq 4$ up to a constant its divergence is the *Cotton tensor* $\mathrm{C} \in \mathcal{T}^{(3,0)}M$

$$\operatorname{div} \mathrm{W}(X,Y,Z) =: -(n-3)\,\mathrm{C}(X,Y,Z). \tag{1.39}$$

The Cotton Tensor defined here is antisymmetric in its last two arguments[1]. Calculation of the divergence of the Weyl tensor gives the following equivalent definition of the Cotton tensor

$$\mathrm{C}(X,Y,Z) = (\nabla_Y \mathrm{P})(Z,X) - (\nabla_Z \mathrm{P})(Y,X). \tag{1.40}$$

The latter formula also defines the Cotton tensor in arbitrary dimension. By using the Cotton tensor to replace derivatives of the Schouten tensor in Equation (1.33) the divergence of the curvature tensor R can be written as

$$0 = \operatorname{div} R(X,Y,Z) + (n-2)\,\mathrm{C}(X,Y,Z) + dJ(Y)g(X,Z) - dJ(Z)g(X,Y). \tag{1.41}$$

The divergence $\operatorname{div}_2 \mathrm{C}$ of the Cotton tensor can now be calculated either by using Equation (1.40) or Equation (1.41).

Corollary 1.1.10. *Let $\{e_i\}$ be an local orthonormal frame at $p \in M$. Then it holds*

$$\begin{aligned}(\operatorname{div}_2 \mathrm{C})(X,Y) = -(\operatorname{div}_3 \mathrm{C})(X,Y) &= (\Delta^\nabla \mathrm{P})(X,Y) - (\operatorname{div}_2(\nabla \mathrm{P}))(Y,X)\\ &= \left(\Delta^\nabla \mathrm{P} + \operatorname{Hess} J\right)(X,Y) + \sum_i \epsilon_i (\mathrm{R}(e_i,Y)\mathrm{P})(e_i,X)\end{aligned} \tag{1.42}$$

Using the Ricci product (Equation (1.20)) the last term can be expressed in terms of the $(3,1)$ curvature tensor R_{vf} by observing $\mathrm{R}(e_i,Y)\mathrm{P}(e_i,X) = (\mathrm{R}_{vf}\cdot\mathrm{P})(e_i,Y,e_i,X)$. In particular the last term equals $\operatorname{tr}_{1,3}\mathrm{R}_{vf}\cdot\mathrm{P}$.

Definition The *Bach tensor* on a Riemannian manifold (M,g) is defined as

$$\mathfrak{B} := -\Delta^\nabla \mathrm{P} + \operatorname{div}_2(\nabla \mathrm{P}) + \operatorname{tr}_{1,3}(\operatorname{tr}_{1,3}\mathrm{P}\otimes \mathrm{W}) \tag{1.43}$$

The signs in this definition are a matter of convention. The definition used here for example has the opposite sign of the Bach tensor that is used in [BJ10, Juh09]. By using Equation (1.42) the Bach tensor also has the following equivalent expressions

$$\begin{aligned}\mathfrak{B} &= \operatorname{tr}_{1,3}(\operatorname{tr}_{1,3}\mathrm{P}\otimes \mathrm{W}) - \operatorname{div}_2 \mathrm{C}\\ &= \operatorname{tr}_{1,3}(\operatorname{tr}_{1,3}\mathrm{P}\otimes \mathrm{W}) + \frac{1}{n-3}\operatorname{div}_2(\operatorname{div}\mathrm{W})\end{aligned} \tag{1.44}$$

It can also be written in a way, such that only derivatives of the Schouten tensor appear

$$\begin{aligned}&\mathfrak{B}(X,Y) =\\ &\quad -\left(\Delta^\nabla \mathrm{P} + \operatorname{Hess} J\right)(X,Y) - n\mathrm{P}(\mathrm{P}^\sharp(Y),X) + \|\mathrm{P}\|_g^2 g(X,Y) + 2\operatorname{tr}_{1,3}(\operatorname{tr}_{1,3}\mathrm{P}\otimes\mathrm{W}),\end{aligned} \tag{1.45}$$

where $\|\mathrm{P}\|_g^2 = \sum_i \epsilon_i \mathrm{P}(\mathrm{P}^\sharp(e_i),e_i)$ for an orthonormal frame $\{e_i\}$. Equation (1.45) is important to parts of this thesis, so a more explicit calculation will be given next. The divergence term $\operatorname{div}_2 \mathrm{C}$ in Equation (1.44) can be removed by using (1.42). This yields

$$\begin{aligned}\mathfrak{B}(X,Y) = &-\Delta^\nabla \mathrm{P} - (\nabla\nabla J)(X,Y) - \sum_i \epsilon_i(\mathrm{R}(e_i,X)\mathrm{P})(e_i,Y) + \sum_i \epsilon_i \mathrm{W}(\mathrm{P}^\sharp(e_i),X,e_i,Y)\\ = &-(\Delta^\nabla \mathrm{P} + \operatorname{Hess} J)(X,Y) + \epsilon_i \mathrm{P}(\mathrm{R}(e_i,X)e_i,Y) + \sum_i \epsilon_i \mathrm{P}(e_i,\mathrm{R}(e_i,X)Y)\\ &+ \sum_i \epsilon_i \mathrm{W}(\mathrm{P}^\sharp(e_i),X,e_i,Y)\\ = &-(\Delta^\nabla \mathrm{P} + \operatorname{Hess} J)(X,Y) - (n-2)\mathrm{P}(\mathrm{P}^\sharp(Y),X) - J\mathrm{P}(X,Y)\\ &+ \sum_i \epsilon_i \mathrm{R}(\mathrm{P}^\sharp(e_i),X,e_i,Y) + \sum_i \epsilon_i \mathrm{W}(\mathrm{P}^\sharp(e_i),X,e_i,Y)\end{aligned}$$

1 Another frequently-used definition for the Cotton tensor is $\operatorname{div} W(X,Y,Z) =: -(n-3)\tilde{\mathrm{C}}(Z,Y,X)$. The two definitions are apparently related by $\mathrm{C}(X,Y,Z) = \tilde{\mathrm{C}}(Z,Y,X)$

$$= -(\Delta^{\nabla}\mathrm{P} + \mathrm{Hess}\, J)(X,Y) - (n-2)\mathrm{P}(\mathrm{P}^{\sharp}(Y),X) - J\mathrm{P}(X,Y) + \sum_i (\mathrm{P} \owedge g)(\mathrm{P}^{\sharp}(e_i)X, e_i, Y) + 2\sum_i \epsilon_i \mathrm{W}(\mathrm{P}^{\sharp}(e_i), X, e_i, Y).$$

Using

$$\sum_i (\mathrm{P} \owedge g)(\mathrm{P}^{\sharp}(e_i), X, e_i, Y) = -2\mathrm{P}(\mathrm{P}^{\sharp}(Y),X) + \|\mathrm{P}\|_g^2 g(X,Y) + J\mathrm{P}(X,Y)$$

then gives Equation (1.45) for the Bach tensor.

Instead of using the Schouten tensor P and its trace J in (1.45), those objects may equivalently be replaced by the Ricci tensor and the scalar curvature

$$\mathfrak{B} = -\frac{1}{n-2}\Delta^{\nabla}\,\mathrm{Ric} - \frac{1}{2(n-1)}\,\mathrm{Hess}\,\tau + \frac{\Delta\tau}{2(n-2)(n-1)} g + \mathcal{R}. \tag{1.46}$$

Here $\mathcal{R}$ is a tensor that does not involve derivatives of the Ricci tensor or the scalar curvature. Its explicit form will not be used in this thesis.

Lemma 1.1.11. *The Cotton and Bach tensors have the following properties*

$$(\mathrm{div}_2\,\mathrm{C})\,(X,Y) = (\mathrm{div}_2\,\mathrm{C})\,(Y,X) \tag{1.47}$$
$$0 = \mathrm{div}\,\mathrm{C} \tag{1.48}$$
$$0 = \mathrm{C}(X,Y,Z) + \mathrm{C}(Y,Z,X) + \mathrm{C}(Z,X,Y) \tag{1.49}$$
$$0 = \mathfrak{B}(X,Y) - \mathfrak{B}(Y,X). \tag{1.50}$$

Namely the divergence of the Cotton tensor is symmetric and vanishes if taken in the first argument, the Cotton tensor fulfils the first Bianchi identity and the Bach tensor is symmetric. In addition the Cotton tensor ist totally trace-free due to the same property of the Weyl tensor.

A proof is given in the appendix.

Lemma 1.1.12. *For the Weyl tensor the following Bianchi equation holds*

$$\begin{aligned}\mathcal{B}(\nabla\,\mathrm{W})(X,Y,Z,U,V) = {} & \mathrm{C}(V,X,Y)g(Z,U) + \mathrm{C}(V,Y,Z)g(X,U) + \mathrm{C}(V,Z,X)g(Y,U) \\ & - \mathrm{C}(U,X,Y)g(Z,V) - \mathrm{C}(U,Y,Z)g(X,V) - \mathrm{C}(U,Z,X)g(Y,V),\end{aligned} \tag{1.51}$$

where $\mathcal{B}(\nabla\,\mathrm{W})(X,Y,Z,U,V) = (\nabla\,\mathrm{W})(X,Y,Z,U,V) + (\nabla\,\mathrm{W})(Z,X,Y,U,V) + (\nabla\,\mathrm{W})(Y,Z,X,U,V)$*. The Bochner Laplacian acting on the Weyl tensor can be written as*

$$\begin{aligned}(\Delta^{\nabla}\,\mathrm{W})(Y,Z,U,V) = {} & -\left(\left(\Delta^{\nabla}\mathrm{P} + \mathrm{Hess}\,J\right) \owedge g\right)(U,V,Y,Z) \\ & - (\nabla\,\mathrm{C})(U,V,Y,Z) + (\nabla\,\mathrm{C})(V,U,Y,Z) \\ & - (n-3)(\nabla\,\mathrm{C})(Y,Z,U,V) + (n-3)(\nabla\,\mathrm{C})(Z,Y,U,V) \\ & - \sum_i \epsilon_i (\mathrm{R}(e_i,Z)\,\mathrm{W})(e_i,Y,U,V) + \sum_i \epsilon_i (\mathrm{R}(e_i,Y)\,\mathrm{W})(e_i,Z,U,V) \\ & + \sum_i \epsilon_i\,(\mathrm{R}(e_i,Y)\mathrm{P})\,(e_i,V)g(U,Z) + \sum_i \epsilon_i\,(\mathrm{R}(e_i,Z)\mathrm{P})\,(e_i,U)g(V,Y) \\ & - \sum_i \epsilon_i\,(\mathrm{R}(e_i,Z)\mathrm{P})\,(e_i,V)g(U,Y) - \sum_i \epsilon_i\,(\mathrm{R}(e_i,Y)\mathrm{P})\,(e_i,U)g(V,Z).\end{aligned} \tag{1.52}$$

Proof: Using the definition of the Kulkarni-Nomizu product one has

$$\begin{aligned}\mathcal{B}(\nabla\mathrm{P} \owedge g)(X,Y,Z,U,V) &= (\nabla\mathrm{P} \owedge g)(X,Y,Z,U,V) + (\nabla\mathrm{P} \owedge g)(Y,Z,X,U,V) \\ &\quad + (\nabla\mathrm{P} \owedge g)(Z,X,Y,U,V) \\ &= \nabla\mathrm{P}(X,Y,U)g(Z,V) + \nabla\mathrm{P}(X,Z,V)g(Y,U) \\ &\quad - \nabla\mathrm{P}(X,Y,V)g(Z,U) - \nabla\mathrm{P}(X,Z,U)g(Y,V) \\ &\quad + \nabla\mathrm{P}(Z,X,U)g(Y,V) + \nabla\mathrm{P}(Z,Y,V)g(X,U) \\ &\quad - \nabla\mathrm{P}(Z,X,V)g(Y,U) - \nabla\mathrm{P}(Z,Y,U)g(X,V) \\ &\quad + \nabla\mathrm{P}(Y,Z,U)g(X,V) + \nabla\mathrm{P}(Y,X,V)g(Z,U) \\ &\quad - \nabla\mathrm{P}(Y,Z,V)g(X,U) - \nabla\mathrm{P}(Y,X,U)g(Z,V)\end{aligned}$$

$$
\begin{aligned}
= \ & \mathrm{C}(U,Y,X)g(Z,V)+\mathrm{C}(V,X,Y)g(Z,U)+\mathrm{C}(U,X,Z)g(Y,V)\\
&+\mathrm{C}(V,Z,X)g(Y,U)+\mathrm{C}(U,Z,Y)g(X,V)+\mathrm{C}(V,Y,Z)g(X,U)
\end{aligned}
$$

and the first claim follows. For the second claim one first calculates

$$
\begin{aligned}
(\nabla\mathcal{B}(\nabla\,\mathrm{W}))(A,B,Y,Z,U,V) &= (\nabla\nabla\,\mathrm{W})(A,B,Y,Z,U,V)+(\nabla\nabla\,\mathrm{W})(A,Z,B,Y,U,V)\\
&\quad -(\nabla\nabla\,\mathrm{W})(A,Y,B,Z,U,V)\\
&= (\nabla\nabla\,\mathrm{W})(A,B,Y,Z,U,V)+(\nabla\nabla\,\mathrm{W})(Z,A,B,Y,U,V)\\
&\quad -(\nabla\nabla\,\mathrm{W})(Y,A,B,Z,U,V)\\
&\quad +(\mathrm{R}(A,Z)\,\mathrm{W})(B,Y,U,V)-(\mathrm{R}(A,Y)\,\mathrm{W})(B,Z,U,V)
\end{aligned}
$$

such that

$$
\begin{aligned}
-\left(\operatorname{div}\mathcal{B}(\nabla\,\mathrm{W})\right)(Y,Z,U,V) = &-(\Delta^{\nabla}\,\mathrm{W})(Y,Z,U,V)\\
&+(n-3)(\nabla\,\mathrm{C})(Z,Y,U,V)-(n-3)(\nabla\,\mathrm{C})(Y,Z,U,V)\\
&-\sum_i \epsilon_i(\mathrm{R}(e_i,Z)\,\mathrm{W})(e_i,Y,U,V)+\sum_i \epsilon_i(\mathrm{R}(e_i,Y)\,\mathrm{W})(e_i,Z,U,V)
\end{aligned}
$$

On the other hand calculating the divergence of the first equation and using Equations (1.48) and (1.42) give

$$
\begin{aligned}
\left(\operatorname{div}\mathcal{B}(\nabla\,\mathrm{W})\right)(Y,Z,U,V) &= (\operatorname{div}_2\mathrm{C})(V,Y)g(U,Z)+(\operatorname{div}_2\mathrm{C})(U,Z)g(V,Y)\\
&\quad -(\operatorname{div}_2\mathrm{C})(V,Z)g(U,Y)-(\operatorname{div}_2\mathrm{C})(U,Y)g(V,Z)\\
&\quad -(\nabla\,\mathrm{C})(U,V,Y,Z)+(\nabla\,\mathrm{C})(V,U,Y,Z)\\
&= -\left(\left(\Delta^{\nabla}\mathrm{P}+\operatorname{Hess}J\right)\owedge g\right)(U,V,Y,Z)\\
&\quad -(\nabla\,\mathrm{C})(U,V,Y,Z)+(\nabla\,\mathrm{C})(V,U,Y,Z)\\
&\quad +\sum_i \epsilon_i\left(\mathrm{R}(e_i,Y)\mathrm{P}\right)(e_i,V)g(U,Z)+\sum_i \epsilon_i\left(\mathrm{R}(e_i,Z)\mathrm{P}\right)(e_i,U)g(V,Y)\\
&\quad -\sum_i \epsilon_i\left(\mathrm{R}(e_i,Z)\mathrm{P}\right)(e_i,V)g(U,Y)-\sum_i \epsilon_i\left(\mathrm{R}(e_i,Y)\mathrm{P}\right)(e_i,U)g(V,Z)
\end{aligned}
$$

■

Proposition 1.1.13. *For the Cotton tensor the following equation holds*

$$
\begin{aligned}
(\Delta^{\nabla}\mathrm{C})(X,Y,Z) = &\left(\nabla\left(\Delta^{\nabla}\mathrm{P}+\operatorname{Hess}J\right)\right)(Y,Z,X)-\left(\nabla\left(\Delta^{\nabla}\mathrm{P}+\operatorname{Hess}J\right)\right)(Z,Y,X)\\
&+2\sum_i \epsilon_i\left(\mathrm{R}(e_i,Z)\nabla\mathrm{P}\right)(e_i,Y,X)-2\sum_i \epsilon_i\left(\mathrm{R}(e_i,Y)\nabla\mathrm{P}\right)(e_i,Z,X)\\
&+(n-2)\,\mathrm{C}(\mathrm{P}^{\sharp}(X),Y,Z)+(n-2)\,\mathrm{C}(Y,\mathrm{P}^{\sharp}(Z),X)-(n-2)\,\mathrm{C}(Z,\mathrm{P}^{\sharp}(Y),X)\\
&+g(X,Y)dJ(\mathrm{P}^{\sharp}(Z))-g(X,Z)dJ(\mathrm{P}^{\sharp}(Y))+\mathrm{P}(X,Z)dJ(Y)-\mathrm{P}(X,Y)dJ(Z)\\
&+\mathrm{R}(\operatorname{grad}J,X,Y,Z).
\end{aligned}
\tag{1.53}
$$

Proof: A short proof is gained by the usage of the Ricci product introduced before. Alternatively one may calculate the result by using p-synchronous vector fields. Using $(\nabla\nabla T)(X,Y,\dots)=$

$(\nabla\nabla T)(Y,X,\dots)+(\mathrm{R}_{vf}\cdot T)(X,Y,\dots)$ for a tensor T twice, one can commute the arguments of $\nabla\nabla\nabla\mathrm{P}$. Hence

$$\begin{aligned}
-\left(\Delta^{\nabla}\mathrm{C}\right)(X,Y,Z) &= \sum_i \epsilon_i\,(\nabla\nabla\nabla\mathrm{P})\,(e_i,e_i,Y,Z,X)-\sum_i\epsilon_i\,(\nabla\nabla\nabla\mathrm{P})\,(e_i,e_i,Z,Y,X)\\
&= \sum_i \epsilon_i\,(\nabla\nabla\nabla\mathrm{P})\,(e_i,Y,e_i,Z,X)-\sum_i\epsilon_i\,(\nabla\nabla\nabla\mathrm{P})\,(e_i,Z,e_i,Y,X)\\
&\quad+\sum_i\epsilon_i\left(\nabla\left(\mathrm{R}_{vf}\cdot\mathrm{P}\right)\right)(e_i,e_i,Y,Z,X)-\sum_i\epsilon_i\left(\nabla\left(\mathrm{R}_{vf}\cdot\mathrm{P}\right)\right)(e_i,e_i,Z,Y,X)\\
&\overset{(1.24)}{=} \sum_i \epsilon_i\,(\nabla\nabla\nabla\mathrm{P})\,(Y,e_i,e_i,Z,X)-\sum_i\epsilon_i\,(\nabla\nabla\nabla\mathrm{P})\,(Z,e_i,e_i,Y,X)\\
&\quad+\sum_i\epsilon_i\,(\mathrm{R}(e_i,Y)\nabla\mathrm{P})\,(e_i,Z,X)-\sum_i\epsilon_i\,(\mathrm{R}(e_i,Z)\nabla\mathrm{P})\,(e_i,Y,X)\\
&\quad+\sum_i\epsilon_i\left(\left(\nabla\,\mathrm{R}_{vf}\right)(e_i,e_i,Y)\cdot\mathrm{P}\right)(Z,X)+\sum_i\epsilon_i\,(\mathrm{R}(e_i,Y)\nabla\mathrm{P})\,(e_i,Z,X)\\
&\quad-\sum_i\epsilon_i\left(\left(\nabla\,\mathrm{R}_{vf}\right)(e_i,e_i,Z)\cdot\mathrm{P}\right)(Y,X)-\sum_i\epsilon_i\,(\mathrm{R}(e_i,Z)\nabla\mathrm{P})\,(e_i,Y,X)\\
&\quad+(\nabla\nabla dJ)\,(Z,Y,X)-(\nabla\nabla dJ)\,(Y,Z,X)-(\mathrm{R}(Z,Y)dJ)\,(X).
\end{aligned}$$

The last line is zero but for example $(\nabla\nabla dJ)\,(Z,Y,X)$ is the same as $(\nabla\,\mathrm{Hess}\,J)\,(Z,Y,X)$ and in the end will be part of the first term in Equation (1.53). Now using Equation (1.41) for the divergence of the curvature tensor yields

$$\sum_i\epsilon_i\,(\nabla\,\mathrm{R})\,(e_i,e_i,U)=(n-2)\,\mathrm{C}^{\sharp_2}(U,\cdot,\cdot)+\mathrm{grad}\,J\otimes U^{\flat}-U\otimes dJ,\tag{1.54}$$

Hence one finds

$$\begin{aligned}
\sum_i\epsilon_i\left(\left(\nabla\,\mathrm{R}_{vf}\right)(e_i,e_i,U)\cdot\mathrm{P}\right)(V,W) &= -\sum_i\epsilon_i\mathrm{P}\left(\left(\nabla\,\mathrm{R}_{vf}\right)(e_i,e_i,U)\cdot V,W\right)\\
&\quad-\sum_i\epsilon_i\mathrm{P}\left(V,\left(\nabla\,\mathrm{R}_{vf}\right)(e_i,e_i,U)\cdot W\right)\\
&\overset{(1.54)}{=} -(n-2)\,\mathrm{C}(U,\mathrm{P}^{\sharp}(W),V)-(n-2)\,\mathrm{C}(U,\mathrm{P}^{\sharp}(V),W)\\
&\quad-g(U,V)\mathrm{P}(\mathrm{grad}\,J,W)+dJ(V)\mathrm{P}(U,W)\\
&\quad-g(U,W)\mathrm{P}(\mathrm{grad}\,J,V)+dJ(W)\mathrm{P}(U,V)\\
&= -(n-2)\,\mathrm{C}(U,\mathrm{P}^{\sharp}(W),V)-(n-2)\,\mathrm{C}(U,\mathrm{P}^{\sharp}(V),W)\\
&\quad-g(U,V)dJ\left(\mathrm{P}^{\sharp}(W)\right)+dJ(V)\mathrm{P}(U,W)\\
&\quad-g(U,W)dJ(\mathrm{P}^{\sharp}(V))+dJ(W)\mathrm{P}(U,V)
\end{aligned}$$

and thus

$$\begin{aligned}
-\left(\Delta^{\nabla}\mathrm{C}\right)(X,Y,Z) = &-(\nabla\,(\Delta+\mathrm{Hess}\,J))\,(Y,Z,X)+(\nabla\,(\Delta+\mathrm{Hess}\,J))\,(Z,Y,X)+dJ(\mathrm{R}(Z,Y)X)\\
&+2\sum_i\epsilon_i\,(\mathrm{R}(e_i,Y)\nabla\mathrm{P})\,(e_i,Z,X)-2\sum_i\epsilon_i\,(\mathrm{R}(e_i,Z)\nabla\mathrm{P})\,(e_i,Y,X)\\
&-(n-2)\,\mathrm{C}(Y,\mathrm{P}^{\sharp}(X),Z)-(n-2)\,\mathrm{C}(Y,\mathrm{P}^{\sharp}(Z),X)\\
&+(n-2)\,\mathrm{C}(Z,\mathrm{P}^{\sharp}(X),Y)+(n-2)\,\mathrm{C}(Z,\mathrm{P}^{\sharp}(Y),X)\\
&-g(Y,Z)dJ\left(\mathrm{P}^{\sharp}(X)\right)+dJ(Z)\mathrm{P}(Y,X)-g(Y,X)dJ(\mathrm{P}^{\sharp}(Z))+dJ(X)\mathrm{P}(Y,Z)\\
&+g(Z,Y)dJ\left(\mathrm{P}^{\sharp}(X)\right)-dJ(Y)\mathrm{P}(Z,X)+g(Z,X)dJ(\mathrm{P}^{\sharp}(Y))-dJ(X)\mathrm{P}(Z,Y).
\end{aligned}$$

Using the Bianchi identity for the Cotton tensor then gives the result. ■

The tensor given in Equation (1.54) also is the divergence of the $(3,1)$-curvature tensor and will be denoted

$$\mathrm{div}\,\mathrm{R}(U):=-\sum_i\epsilon_i\,(\nabla\,\mathrm{R})\,(e_i,e_i,U)=-(n-2)\,\mathrm{C}^{\sharp_2}(U,\cdot,\cdot)-\mathrm{grad}\,J\otimes U^{\flat}+U\otimes dJ.\tag{1.55}$$

Using this notation and the definition of the dot-product one can find a short notation for commuting the Bochner Laplacian Δ^{∇} with the Levi-Civita connection ∇.

Lemma 1.1.14. *Let $T \in \mathcal{T}^{p,q}M$ be an arbitrary tensor field on (M,g) with $p \geq 1$, then it holds*

$$[\Delta, \nabla]\, T = -2\,\mathrm{tr}_{1,3}\left((\mathrm{R}_{vf} \cdot (\nabla T)\right) + (\mathrm{div}\, \mathrm{R}_{vf}) \cdot T, \tag{1.56}$$

where $\left(\left(\mathrm{div}\,\mathrm{R}_{vf}\right) \cdot T\right)(X, \cdot, \ldots, \cdot) := \left(\left(\mathrm{div}\,\mathrm{R}_{vf}\right)(X) \cdot T\right)(\cdot, \ldots, \cdot)$.

One wouldn't loose generality if one didn't demand the first argument of T to be a vector. In that case the traces on the right-hand side will be contractions. Since it is not needed throughout the thesis only the first case is of interest.
Proof: Let be θ_j be vectors or covectors, $\{e_i\}$ an orthonormal frame and $X \in T_pM$ an arbitrary vector, then

$$\begin{aligned}
(\Delta(\nabla T))\,(X, \theta_1, \ldots, \theta_k) &= -\sum_i \epsilon_i (\nabla\nabla\nabla T)(e_i, e_i, X, \theta_1, \ldots, \theta_k) \\
&= -\sum_i \epsilon_i (\nabla\nabla\nabla T)(e_i, X, e_i, \theta_1, \ldots, \theta_k) \\
&\quad -\sum_i \epsilon_i \left(\nabla(\mathrm{R}_{vf} \cdot T)\right)(e_i, e_i, X, \theta_1, \ldots, \theta_k) \\
&\overset{(1.24)}{=} (\nabla(\Delta T))\,(X, \theta_1, \ldots, \theta_k) - 2\sum_i \epsilon_i \left(\mathrm{R}(e_i, X)(\nabla T)\right)(e_i, \theta_1, \ldots, \theta_k) \\
&\quad -\sum_i \epsilon_i \left((\nabla \mathrm{R}_{vf})(e_i, e_i, X) \cdot T\right)(\theta_1, \ldots, \theta_k).
\end{aligned}$$

Rewriting the last term gives the claim. ■

1.1.1. Vector Fields and Flows

We will first fix the notation for this section. A curve $\gamma : (\alpha, \beta) \subset \mathbb{R} \to M$ is an *integral curve* of a vector field $X \in \mathfrak{X}(M)$ if for all $t \in (\alpha, \beta)$ it holds $\dot{\gamma}(t) = X_{\gamma(t)}$. Provided γ is not extendible to a bigger interval, it is a *maximal integral curve* . For a point $p \in M$ then the maximal integral curve $\gamma : I \to M$ of X with $\gamma(t) = p$ for some $t \in I$ is unique.

The map $\phi : U \to M$ with open subset $U \subset \mathbb{R} \times M$ is a *local flow* of $X \in \mathfrak{X}(M)$, provided $\dot{\phi}(t,x) = X_{\phi(t,x)}$ for all $(t,x) \in U$ and $\phi(0, \cdot) = \mathrm{id}$. The maximal domain $\mathcal{D}$ where the flow can be defined is

$$\mathcal{D} := \bigcup_{p \in M} I_p \times \{p\}, \tag{1.57}$$

where I_p is the domain of the unique maximal integral curve $\gamma_p^X : I_p \to M$ of X with $\gamma(0) = p$. The map $\Phi : \mathcal{D} \ni (t,x) \mapsto \gamma_p^X(t) \in M$ is referred to as the *flow of* X. Properties of the (local) flow $\Phi : \mathcal{D} \to M$ of X are

(i) $\Phi(0,p) = p$ for all $p \in M$,
(ii) $\Phi(t_0 + t_1, p) = \Phi(t_0, \Phi(t_1, p))$ for all $p \in M$ and $t_0, t_1, t_0 + t_1 \in I_p$,
(iii) $\Phi : \mathcal{D} \to M$ is a smooth map,
(iv) $\Phi(t, \cdot)$ is a diffeomorphism where it is defined with inverse $\Phi(t, \cdot)^{-1} = \Phi(-t, \cdot)$ and
(v) the domain $\mathcal{D}$ is open[2].

Repellers and Attractors

Let $p \in M$ be a fixed point of $X \in \mathfrak{X}(M)$, i.e. $X_p = 0$. Then p is an *attractor* of X if every neighbourhood $\tilde{U}$ of p contains a neighbourhood $U \subset \tilde{U}$, such that every maximal integral curve γ of X with $\gamma(0) \in U$ is defined on the hole interval $I = [0, \infty)$, $\gamma(t) \in \tilde{U}$ for all $t \in I$ and $\gamma(t) \xrightarrow[t \to \infty]{} p$. p is called a *repeller* of X, if with the same requirements γ is defined on $I = (-\infty, 0]$ and $\gamma(t) \xrightarrow[t \to -\infty]{} p$. Equivalently p is an repeller of X, if it is an attractor of $-X$.

2 Proofs may be found for example in [GMK75, Chapter 8.6ff].

Definition 1.1.15. Now let $p \in M$ be an attractor of $X \in \mathfrak{X}(M)$ and U a neighbourhood of p. U will be called an *attracting neighbourhood of p with respect to X* if all maximal integral curves $\gamma : (\alpha, \beta)$ of X with $\gamma(0) \in U$ are defined at least at $I = [0, \infty)$, $\gamma(t) \in U$ for all $t \in I$ and $\gamma(t) \to p$ for $t \to \infty$. U will be called an *repelling neighbourhood* if it is an attracting neighbourhood of p with respect to $-X$.

If X is a vector field on a subset of $\mathbb{R}^n$, instead of X_p the notation $X(p)$ will be used for the value of X at $p \in \mathbb{R}^n$. A basic but important lemma is the following.

Lemma 1.1.16. *Let $X : \mathbb{R}^n \supset U \to \mathbb{R}^n$ be a smooth vector field, such that for at least one component $\left|X^k(x)\right| > \delta > 0$ for all $x \in U$. Let $p \in U$ be a point and consider $\epsilon > 0$ such that the open ball $B_{2\epsilon}(p)$ is a subset of U. Let be $B := B_\epsilon(p) \subset U$, then every maximal integral curve starting within B ($\gamma(t) \in B$ for some $t \in \mathbb{R}$) will leave B within finite time in both directions.*

The lemma is a consequence of existence and uniqueness results for first-order ordinary differential equations (see the appendix for details). Attractors and repellers are characterised by the following theorem.

Theorem 1.1.17. *[Poincaré-Lyapunov] Consider $\mathcal{U} \subset \mathbb{R}^n$ to be an open neighbourhood of 0, $X \in C^1(\mathcal{U}, \mathbb{R}^n)$ a vector field and $0 \in \mathcal{U}$ a fixed point of X. If the Jacobian*

$$X'(0) := (\partial_1 X(0) \cdots \partial_n X(0)) \tag{1.58}$$

has only eigenvalues with negative real part, then 0 is an attractor. If all the eigenvalues have positive real part, then 0 is an repeller.

A proof can for example be found in [Kön04, Chapter 4.5].

Lemma 1.1.18. *Let $\mathcal{U} \subset \mathbb{R}^n$ be an open neighbourhood of 0, $X \in C^1(\mathcal{U}, \mathbb{R}^n)$ a smooth vector field and $0 \in \mathcal{U}$ a fixed point of X.*

i If $X'(0)$ is negative definite, i.e. $\forall V \in \mathbb{R}^n : \langle X' \cdot V, V\rangle < 0$, then there exists an $r > 0$ such that all open balls $B_{\tilde{r}}(0)$ with $\tilde{r} \le r$ are attracting neighbourhoods of 0. Such balls will be called attracting balls.

i If $X'(0)$ is positive definite, then there is an $r > 0$, such that all open balls $B_{\tilde{r}}(0)$ with $\tilde{r} \le r$ are repelling neighbourhoods. Such balls will be called repelling balls.

Proof: Let $A := X'(0)$ be the Jacobian of X in 0 and let

$$\mu_0 := \max \{\langle Ax, x\rangle \mid \|x\| = 1\}.$$

μ_0 is well defined due to continuity of $\langle Ax, x\rangle$ and $\mu_0 < 0$, since A is negative definite. Hence $\langle Ax, x\rangle \;< \mu_0 \|x\|^2$ for all $x \in \mathbb{R}^n \setminus \{0\}$. Taylor's theorem gives

$$X(x) = Ax + R(x)x,$$

with $R(x) = \mathcal{O}(\|x\|)$. Hence $r > 0$ can be chosen such that for all x in the open ball $B_{2r}(0) \subset \mathcal{U}$

$$\langle R(x)x, x\rangle \le -\frac{\mu_0}{2}\langle x, x\rangle = \frac{|\mu_0|}{2}\langle x, x\rangle.$$

Hence for $x \in B_{2r}(0)$

$$\langle X(x), x\rangle < \frac{\mu_0}{2}\langle x, x\rangle.$$

Now consider $\gamma : (-\alpha, \beta) \to \mathcal{U}$ to be a maximal integral curve with $\gamma(0) \in B_r(0)$. Then it suffices to show $\|\gamma(t)\| \le \|\gamma(0)\|$ for all $t \in [0, \beta)$. Let $f : [0, \beta) \to \mathbb{R}^+$ be defined by $f(t) := \langle \gamma(t), \gamma(t)\rangle$, then $\dot{f}(t) = 2\langle \dot{\gamma}(t), \gamma(t)\rangle = 2\langle X(\gamma(t)), \gamma(t)\rangle$ and therefore $\dot{f}(0) < \mu_0$. Assumed there is a $\tilde{t} \in (0, \beta)$ with $\gamma(\tilde{t}) \in B_r(0) \setminus \overline{B_{\|\gamma(0)\|}}(0) \subset B_{2r}(0)$. Then there is an interval $\tilde{I} = [\tilde{\alpha}, \tilde{\beta}] \subset [0, \tilde{t}]$ such that $f(\tilde{\alpha}) = \|\gamma(\tilde{\alpha})\|^2 = \|\gamma(0)\|^2$, $f(\tilde{\beta}) = \left\|\gamma(\tilde{\beta})\right\|^2 = \|\gamma(\tilde{t})\|^2$ and $\gamma(t) \in B_{\|\gamma(\tilde{t})\|}(0) \setminus \overline{B_{\|\gamma(0)\|}(0)} \subset$

$B_{2r}(0)$ for all $t \in (\tilde{\alpha}, \tilde{\beta})$. In particular $f(\tilde{\beta}) > f(\tilde{\alpha})$. From the mean value theorem, there is a $t^\star \in (\tilde{\alpha}, \tilde{\beta})$ such that

$$\begin{aligned} \dot{f}(t^\star) &= \frac{f(\tilde{\beta}) - f(\tilde{\alpha})}{\tilde{\beta} - \tilde{\alpha}} \\ &> 0. \end{aligned}$$

Now since $t^\star$ is an element of $\tilde{I}$, one also has $\gamma(t^\star) \in B_{2r}(0)$ and hence $\dot{f}(t^\star) = \langle X(\gamma(t^\star)), \gamma(t^\star)\rangle < \mu_0 < 0$ which is a contradiction. Hence $B_r(0)$ is an attracting neighbourhood of 0.

The proof for the existence of a repelling neighbourhood follows from the fact, that it is attracting for $-X$. ■

Lemma 1.1.19. *Let $X \in \mathfrak{X}(M)$ be a smooth vector field on M and let $p \in M$ be an attractor or repeller of X. Let $\gamma : (\alpha, \beta) \to M$ be a maximal integral curve of X repelled or attracted by p, i.e. either $\gamma(t) \to p$ or $\gamma(-t) \to p$ for $t \to \infty$. Then either γ is a constant curve or it is not closed, i.e. for all $t_1 \neq t_2 \in (\alpha, \beta)$ one also has $\gamma(t_1) \neq \gamma(t_2)$.*

Proof: Assume there are $t_1, t_2 \in (\alpha, \beta)$ such that $\gamma(t_1) = \gamma(t_2) =: q$ and denote $\Delta := t_2 - t_1$. Then $\tilde{\gamma}(t) := \gamma(\Delta + t)$ fulfils $\tilde{\gamma}(t_1) = q$ and $\dot{\tilde{\gamma}}(t_1) = X_{\tilde{\gamma}(t_1)} = X_{\gamma(t_2)} = X_{\gamma(t_1)}$. Hence γ and $\tilde{\gamma}$ coincide by uniqueness of integral curves. The limits $t \to \infty$ and $t \to -\infty$ exist if and only if γ is constant, in particular $\gamma \equiv p$. ■

1.2 CAUSAL STRUCTURE OF LORENTZIAN MANIFOLDS

This section will focus on pseudo-Riemannian manifolds (M, g) of dimension n and signature $(1, n-1)$. Such a signature is also called Lorentzian signature. Basically this section follows [O'N83] and provides basic definition and concepts concerning the causal structure in Lorentzian geometry.

Definition First the causal character of a vector is defined. Consider $p \in M$, a non-vanishing tangent vector $X \in T_pM$ is called

$$\begin{array}{lcl} \textit{timelike} & \Leftrightarrow & g(X,X) < 0 \\ \textit{null} & \Leftrightarrow & g(X,X) = 0 \\ \textit{spacelike} & \Leftrightarrow & g(X,X) > 0 \\ \textit{causal} & \Leftrightarrow & g(X,X) \leq 0. \end{array}$$

A embedded submanifold N of (M, g) may not have a induced metric that is degenerate. In that case, the null vector that defines the degenerate direction of g_p in T_pN also is called *isotropic vector*. Any non-trivial *totally isotropic vector subspace*[3] $\mathcal{V} \subset T_pM$ in Lorentzian signature is of dimension 1. An orthonormal frame $\{e_i\}$ of T_pM will always refer to a basis with indices running from 0 to $n-1$ and for which $g_p(e_i, e_j) = \eta_{ij}$ where η is the Minkowski metric. Hence the timelike direction is given by e_0

Remark 1.2.1. Let be e_i an orthonormal frame in T_pM with $g(e_0, e_0) = -1$. Then the frame $\{n_i\}$, defined by

$$n_i = e_0 + e_i \quad \text{for } i > 0 \tag{1.59}$$

$$n_0 = \frac{n-1}{n-2}e_0 - \frac{\sqrt{n-1}}{n-2}(e_1 + \cdots + e_{n-1}). \tag{1.60}$$

is a basis of null vectors with the property $g(n_i, n_j) = \delta_{ij} - 1$, where δ_{ij} is the Kronecker delta. Such a basis will be called *null basis*.

3 A subspace $\mathcal{V} \subset T_pM$ will be called totally isotropic if $g_p(X,Y) = 0$ for all $X, Y \in \mathcal{V}$.

Regarding to the causal character of a vector one defines the following subsets of the tangent space.

Definition Let (M, g) be a Lorentzian manifold, $p \in M$. Then

$$\mathfrak{T}_p M := \left\{ X \in T_p M \quad | \quad \|X\|^2 := g(X, X) < 0 \right\} \tag{1.61}$$
$$\mathfrak{C}_p M := \left\{ X \in T_p M \quad | \quad \|X\|^2 := g(X, X) = 0, \right\} \tag{1.62}$$
$$\mathfrak{K}_p M := \mathfrak{T}_p M \cup \mathfrak{C}_p M. \tag{1.63}$$

The subset of all timelike vectors $\mathfrak{T}_p M$ is called *time cone* at p, while the subset of all null vectors $\mathfrak{C}_p M$ is called *null cone* at p. To distinguish them from a similar definition that is given later, they will be referred to as tangent time cone and tangent null cone. The union $\mathfrak{K}_p M$ of both sets is called *(tangent) causal cone* .

For keeping notations short, $g(X, X)$ is shorten to $\|X\|^2$ if there is no confusion about the metric in use [4]. Otherwise it will be made explicit by writing $\|X\|_g^2$. The causal character of curves in (M, g) is specified in a similar way.

Definition A smooth curve $\gamma : I \to M$, with $I \subset \mathbb{R}$ being an interval, is called

$$\begin{array}{lll} \textit{timelike} & \Leftrightarrow & \|\dot\gamma(t)\|^2 < 0 \quad \forall t \in I \\ \textit{null} & \Leftrightarrow & \|\dot\gamma(t)\|^2 = 0 \quad \forall t \in I \\ \textit{spacelike} & \Leftrightarrow & \|\dot\gamma(t)\|^2 > 0 \quad \forall t \in I \\ \textit{causal} & \Leftrightarrow & \|\dot\gamma(t)\|^2 \leq 0 \quad \forall t \in I \end{array}$$

if in addition it has nowhere vanishing tangent vector $\dot\gamma(t)$. Assume γ to be smooth only piecewise and $V : I \to TM$ a not necessarily continuous map with $V(t) \in T_{\gamma(t)} M$ being a timelike vector for all $t \in I$. Then γ usually is called timelike (null, causal), if it is timelike (null, causal) on its smooth parts and if in addition for all $t \in I$

$$g_{\gamma(t)}\left(\dot\gamma(t^-), V(t)\right) \cdot g_{\gamma(t)}\left(\dot\gamma(t^+), V(t)\right) > 0.$$

Here $\dot\gamma(t^+) := \lim_{s \searrow t} \dot\gamma(s)$ and $\dot\gamma(t^-) := \lim_{s \nearrow t} \dot\gamma(s)$.

Definition A Lorentzian manifold (M, g) is said to be *time-orientable* if it admits a non-vanishing timelike vector field $O_T \in \mathfrak{X}(M)$. The *time-orientation* with respect to that vector field then is the decomposition

$$\mathfrak{T}_p M = \mathfrak{T}_p^{\uparrow} M \cup \mathfrak{T}_p^{\downarrow} M \tag{1.64}$$

at each point $p \in M$, with $\mathfrak{T}_p^{\uparrow} M := \{X \in \mathfrak{T}_p M \,|\, g(X, O_T) < 0\}$ being the set of *future-directed* timelike vectors at p, while $\mathfrak{T}_p^{\downarrow} M := \{X \in \mathfrak{T}_p M \,|\, g(X, O_T) > 0\}$ is the set of *past-directed* timelike vectors at p. A similar notation is used for the null and the causal cone.

$$\mathfrak{C}_p M \setminus \{0\} = \mathfrak{C}_p^{\uparrow} M \cup \mathfrak{C}_p^{\downarrow} M. \tag{1.65}$$
$$\mathfrak{K}_p M \setminus \{0\} = \mathfrak{K}_p^{\uparrow} M \cup \mathfrak{K}_p^{\downarrow} M. \tag{1.66}$$

Consequently, if (M, g) is time-oriented with respect to $O_T \in \mathfrak{X}(M)$, then a causal vector $X \in T_p M$ or a causal curve $\gamma : I \to M$ with tangent vector $\dot\gamma(t)$ is called

$$\begin{array}{lll} \textit{future-directed} & \Leftrightarrow & \begin{array}{l} g(\dot\gamma, O_T) < 0 \quad \forall t \in I \\ \text{or } g(X, O_T) < 0 \end{array} \\ \textit{past-directed} & \Leftrightarrow & \begin{array}{l} g(\dot\gamma, O_T) > 0 \quad \forall t \in I \\ \text{or } g(X, O_T) > 0 \end{array} \end{array}$$

Definition [causality relation] Let (M, g) be an time-oriented Lorentzian manifold, $U \subset M$ and $x, y \in M$. Then the following notation is used

4 Recall that despite the notation, this is not a norm.

$$
\begin{aligned}
x \lll_U y \quad &\Leftrightarrow \quad \begin{cases} \exists\gamma : [0, \epsilon \neq 0] \to U \quad \text{smooth on } (0, \epsilon) \\ \gamma(0) = x,\ \gamma(\epsilon) = y \\ \gamma \text{ future-directed, timelike curve} \end{cases} \\
x <_U y \quad &\Leftrightarrow \quad \begin{cases} \exists\gamma : [0, \epsilon \neq 0] \to U \quad \text{smooth on } (0, \epsilon) \\ \gamma(0) = x,\ \gamma(\epsilon) = y \\ \gamma \text{ future-directed, causal curve} \end{cases} \\
x \leq_U y \quad &\Leftrightarrow \quad x <_U y \text{ or } x = y.
\end{aligned}
$$

If $U = M$ the subscript U is omitted and the notation is $x \lll y$, $x < y$ and $x \leq y$. Following the notation in [HE73] special subsets characterising the causal structure of a Lorentzian manifold will now be defined. To this end let be $S, T \subset M$. The *chronological future/past* of S relative to T is

$$I^+(S, T) := \{x \in T \mid \exists y \in S : y \lll_T x\} \tag{1.67}$$
$$I^-(S, T) := \{x \in T \mid \exists y \in S : y \ggg_T x\}. \tag{1.68}$$

If $S = \{p\}$ is a single point, the chronological sets will be denoted $I^\pm(\{p\}, T) =: I^\pm(p, T)$ and for T being the whole manifold, it will be written as $I^\pm(S, M) =: I^\pm(S)$. The *causal future/past* of S relative to T is

$$J^+(S, T) := (S \cap T) \cup \{x \in T \mid \exists y \in S : y \leq_T x\} \tag{1.69}$$
$$J^-(S, T) := (S \cap T) \cup \{x \in T \mid \exists y \in S : y \geq_T x\}. \tag{1.70}$$

Again short notations will be $J^\pm(p, T)$ for single points and $J^\pm(S)$ if applied to the whole manifold. Now the set of sets

$$\bigcup_{p \in M} \{I^+(p), I^-(p), J^+(p), J^-(p)\}$$

is called the *causal structure* of (M, g). The *future/past horismos* of S relative to T is

$$E^+(S, T) := J^+(S, T) \setminus I^+(S, T) \tag{1.71}$$
$$E^-(S, T) := J^-(S, T) \setminus I^-(S, T) \tag{1.72}$$

Short notations are $E^\pm(p, T)$, $E^\pm(S)$.

Definition Let $\mathcal{V} = \mathcal{V}(p, U)$ be one of the causal sets defined above. The *closure* $\overline{\mathcal{V}}(p, U)$, *interior* $\mathring{\mathcal{V}}(p, U)$ and *boundary* $\partial\mathcal{V}(p, U)$ are the topological quantities taken with respect to the open set U.

Consider $U \subset M$ to be open, then the sets $I^\pm(S, U)$ are open, since if $y \in U$ can be reached by a future- or past-directed timelike curve from S, then it has a sufficiently small neighbourhood that can be reached by a small variation of that curve, without changing its causal character. Moreover one finds for the closure, interior and boundary of the sets defined above

$$\overline{I}^\pm(p, U) = \overline{J}^\pm(p, U) \qquad \mathring{I}^\pm(p, U) = \mathring{J}^\pm(p, U) \qquad \partial I^\pm(p, U) = \partial J^\pm(p, U). \tag{1.73}$$

Furthermore for all points $x \in E^\pm(S, U)$ there exists a future- or past-directed null geodesic $\gamma : I \to M$ with $\gamma(0) \in S$ and $\gamma(\epsilon) = x$ for some $\epsilon \geq 0$. If U is a convex normal neighbourhood of $p \in M$, then $E^\pm(p, U)$ is the union of null geodesics starting in p. As a result one gets for a convex normal neighbourhood U:

$$I^\pm(p, U) = \mathring{J}^\pm(p, U) \qquad\qquad E^\pm(p, U) = \partial J^\pm(p, U). \tag{1.74}$$

Definition Let (M, g) be a time-oriented Lorentzian manifold, $U \subset M$ and $p \in M$. Then the *geodesic future/past null cone* in p relative to U is

$$\mathcal{C}_p^\pm(U) = \{p\} \cup \left\{ x \in U \;\middle|\; \begin{array}{l} \exists\gamma : [0, \epsilon] \to U \text{ null geodesic :} \\ \gamma(0) = p,\ \gamma(\epsilon) = x \end{array} \right\}. \tag{1.75}$$

In particular, if U is a convex normal neighbourhood of p, then the geodesic null cone $\mathcal{C}_p^{\pm}(U)$ in p coincides with the horismos $E^{\pm}(p,U)$ of p.

Definition Let (N,h) be a submanifold of a Lorentzian manifold (M,g) with $\dim(M) > 2$ and h being the bilinear form induced on N by g. Then (N,h) is called

$$\begin{aligned} \textit{spacelike submanifold} &\iff h \text{ is a positive definite metric.} \\ \textit{timelike submanifold} &\iff h \text{ is a Lorentzian metric.} \\ \textit{null submanifold} &\iff h \text{ degenerates.} \end{aligned}$$

Consider N to be a codim $= 1$ manifold. Then the above definitions are equivalent to the existence of a vector $n_p \in T_pM$ for each $p \in N$ such that the tangent space T_pN is orthogonal to n_p, i.e. $g(n_p, V) = 0 \; \forall V \in T_pN$. (N,h) is

$$\begin{aligned} \textit{spacelike} &\iff \|n_p\|^2 < 0 \quad \forall p \in N. \\ \textit{timelike} &\iff \|n_p\|^2 > 0 \quad \forall p \in N. \\ \textit{null} &\iff \|n_p\|^2 = 0 \quad \forall p \in N. \end{aligned}$$

If (N,h) is a null hypersurface, then n_p identifies the *isotropic direction* on T_pN.

From here we will implicitly assume that the considered Lorentzian manifolds are time-oriented.

1.2.1. Geodesics on Lorentzian Manifolds

A curve $\gamma : I \to M$ is a *geodesic*, if $\nabla_{\dot\gamma}\dot\gamma = 0$ along γ. It is a *maximal geodesic*, if its domain I is inextendible. An often used fact is that for each $p \in M$ and $X \in T_pM$ there is a unique maximal geodesic $\gamma_X : I(X) \to M$ such that $\gamma_X(0) = p$ and $\dot\gamma(0) = X$. The interval $I(X)$ depends on the vector X. Moreover the set

$$\mathcal{D}_p := \left\{ X \in T_pM \,|\, 1 \in I(X) \right\} \tag{1.76}$$

is open and star-shaped with respect to the origin $0 \in T_pM$.

Definition Let $\alpha, \beta \in \mathbb{R}^+ \setminus \{0\}$. A geodesic segment $\gamma : [t_0, t_0 + \alpha] \to M$ is *closed* if

$$\gamma(t_0 + \alpha) = \gamma(t_0) \qquad\qquad \dot\gamma(t_0 + \alpha) = \beta\dot\gamma(t_0).$$

It is called *periodic* , if $\beta = 1$.

Assume $\beta < 0$ and consider the geodesic $\eta : [t_0, t_0 - \beta\alpha] \to M$ defined by $\eta(t) := \gamma(\alpha - \frac{t_0}{t} + t_0 + \frac{1}{\beta}t)$. Then $\eta(t_0) = \gamma(t_0 + \alpha) = \gamma(t_0)$ and $\dot\eta(t_0) = \frac{1}{\beta}\dot\gamma(t_0 + \alpha) = \dot\gamma(t_0)$. Therefore $\eta = \gamma$ by uniqueness of geodesics. In particular the choice $\tilde t := t_0 - \frac{\beta}{1-\beta}\alpha \in [t_0, t_0 - \beta\alpha] \cap [t_0, t_0 + \alpha]$ leads to $\frac{1}{\beta}\dot\gamma(\tilde t) = \dot\gamma(\tilde t)$, which is a contradiction, since β was assumed to be negative. In particular this shows the impossibility of values $\beta < 0$.

As a matter of fact due to [O'N83, Proposition 7.13], the maximal geodesic extension of a closed segment is maximal if and only if $\beta = 1$. Moreover following the proof therein one has the following corollary.

Corollary 1.2.2. *Let $\gamma : I \to M$ be a maximal null geodesic extension of a closed geodesic segment such that $\gamma(t_0) = \gamma(t_0 + \alpha)$ and $\dot\gamma(t_0 + \alpha) = \beta\dot\gamma(t_0)$, then*

$$I = \begin{cases} \left(-\infty\,,\, t_0 + \alpha\frac{\beta}{\beta-1}\right) & \textit{for } \beta > 1 \\ \mathbb{R} & \textit{for } \beta = 1 \\ \left(t_0 - \alpha\frac{\beta}{1-\beta}\,,\, \infty\right) & \textit{for } \beta \in (0,1) \end{cases} \tag{1.77}$$

In some situations it is beneficial to have a definition for curves that are geodesics up to reparametrisation.

Definition Let (M, g) be a semi-Riemannian manifold, $I \subset \mathbb{R}$ an interval. A smooth curve $\gamma : I \to M$ is a *pregeodesic* if it has nowhere vanishing differential $d\gamma$ and there exists a smooth function $c : I \to \mathbb{R}$ such that

$$\left(\nabla_{\dot{\gamma}}\dot{\gamma}\right)(t) = c(t)\dot{\gamma}(t). \tag{1.78}$$

In most situation it is much simpler to get pregeodesics instead of geodesics. That often it is sufficient to work with the former is provided by the following lemma.

Lemma 1.2.3. *Let $\gamma : I \to M$ be a pregeodesic satisfying Equation (1.78) for some smooth function c. Then there exists a reparametrisation[5] $h : I' \to I$ such that $\tilde{\gamma} := \gamma \circ h$ is a geodesic in M.*

An outline to the proof is given in [O'N83] of which a detailed elaboration is provided in the appendix. Some of its calculations will be used later. For time- or spacelike pregeodesics the reparametrisation is done by normalising to constant length. The reparametrised curve $\tilde{\gamma}$ then satisfies $\frac{d}{ds}g(\tilde{\gamma}', \tilde{\gamma}') = 0$.

1.2.2. Exponential Map

Consider $p \in M$. The maximal domain $\mathcal{D}_pM \subset T_pM$ defined in Equation (1.76) by maximal geodesics originating in p can be used to define the *exponential map* $\exp_p$ in p. It is

$$\begin{array}{rccc} \exp_p : & \mathcal{D}_pM & \to & M \\ & X & \mapsto & \gamma_X(1), \end{array}$$

where γ_X is the maximal geodesic with initial tangent vector $X \in \mathcal{D}_pM$. The exponential map is a local diffeomorphism in a neighbourhood of the origin $0 \in T_pM$, i.e. there exists a star-shaped neighbourhood $\mathfrak{U} \subset \mathcal{D}_p$ of 0 such that $\exp_p : \mathfrak{U} \to \exp_p(\mathfrak{U})$ is a diffeomorphism [O'N83, Proposition 3.30]. The set $\exp_p(\mathfrak{U})$ is called a *normal neighbourhood* of p. As a consequence there is a normal neighbourhood for each $p \in M$. An open set $\mathcal{U} \subset M$ is called *convex* provided it is a normal neighbourhood of each of its points. A simple important consequence of the existence of a convex neighbourhood for each point in M (see for example [O'N83, Proposition 5.7]) is that geodesics do not "end" within the manifold. The fact will be made a lemma, as it is used later on.

Lemma 1.2.4. *Let $\mathcal{U} \subset M$ be a convex open set and $\gamma : [0, t_0) \to \mathcal{U}$ a geodesic such that the limit $\lim_{t\to t_0} \gamma(t) \in \mathcal{U}$ exists. Then $t_0 < \infty$ and γ admits a geodesic extension beyond t_0.*

Another important property is that locally the exponential map $\exp_p$ is a radial isometry (Gauß lemma), i.e. $g_p(X, W) = g_q\left(\left[d\exp_p\right]_X(X), \left[d\exp_p\right]_X(W)\right)$ for all $X, W \in T_pM$ with the identification $T_X\left(T_pM\right) \simeq T_pM$ and with the notation $q = \exp_p(X) \in U_p$. In case where a geodesic starting at a point p is explicitly written in terms of the exponential map $\exp_p$, a consequence of Corollary 1.2.2 can be formulated as follows.

Lemma 1.2.5. *Let $p \in M$ be a point and $X \in T_pM$. Consider the geodesic defined by $\gamma(t) = \exp_p(tX)$. If γ is a closed geodesic with $\gamma(t_0) = \gamma(t_0 + \alpha)$ and $\dot{\gamma}(t_0 + \alpha) = \beta\dot{\gamma}(t_0))$ for $\alpha, \beta \in \mathbb{R}^+ \setminus \{0\}$ then*

(i) $t_0 < \alpha\frac{\beta}{1-\beta}$ for $\beta \in (0, 1)$ and there is no restriction to t_0 if $\beta \geq 1$.

(ii) $\exists t \in [t_0, t_0 + \alpha]$ with $\gamma(t) = p$.

The restriction to positive values of β is reasonable since by Corollary 1.2.2 the set of geodesics with $\beta \leq 0$ is empty. The proof is left to the appendix just as the proof for the following proposition.

5 A reparametrisation is a smooth, surjective map with nowhere vanishing tangent vector

Proposition 1.2.6. *Consider (M,g) to be a time-oriented Lorentzian manifold, $p \in M$, $\mathcal{U}$ a normal neighbourhood of p and $\mathcal{C}_p(\mathcal{U})$ the geodesic null cone in p. Furthermore let $\mathcal{N} \in \mathfrak{X}(\mathcal{U})$ be a vector field on U with the following properties:*

$$\begin{aligned} \mathcal{N}|_{\mathcal{U}\setminus\{p\}} &\neq 0 \\ \|\mathcal{N}\|^2\Big|_{\mathcal{C}_p(\mathcal{U})} &= 0. \\ \mathcal{N}_x \in T_x\mathcal{C}_p(\mathcal{U}) & \qquad \textit{for} \quad x \in \mathcal{C}_p(\mathcal{U}) \end{aligned}$$

In particular $\mathcal{N}$ defines the isotropic direction on the tangent space of $\mathcal{C}_p(\mathcal{U}) \setminus \{p\}$. Let $T \in \mathcal{T}^{(p,0)}M$ be a tensor which is annihilated by $\mathcal{N}$ along the null cone $\mathcal{C}_p(U) \setminus \{p\}$, i.e.

$$T(\mathcal{N}, \cdot, \dots, \cdot) =|_{\mathcal{C}_p(U)} 0.$$

Then T vanishes at p

$$T_p = 0.$$

1.2.3. Jacobi Fields and Causality Theorem

Null, time and causal cone of a point locally can be related to the image of the exponential map at that point. The main theorems dealing with this issue will be summarised in this section. A more extended analysis can be found in [O'N83] and [HE73].

First some notations will be fixed. Let $N \subset M$ be some submanifold of (M,g) with non-degenerated induced metric h. The generic projections of the tangent space or normal space at a point $p \in N$ are denoted $\pi_p^N : T_pM \to T_pN$ and $\pi_p^\perp : T_pM \to T_pN^\perp$. A geodesic $\gamma : I \to M$ is said to be *normal* to N if $\gamma(0) \in N$ and $\dot\gamma(0) \in TN^\perp$.

Definition The *normal connection* $\nabla^\perp : \mathfrak{X}(N)^\perp \to \Gamma\left(T^*N \otimes TN^\perp\right)$ of N acting on sections of the normal bundle is denoted by

$$\nabla_X^\perp Y := \pi^\perp \nabla_X^g Y, \tag{1.79}$$

where $X \in \mathfrak{X}(N)$ and $Y \in \mathfrak{X}(N)^\perp$. The difference of the Levi-Civita and the normal connection is denoted by

$$\tilde{\mathrm{II}}(X,Y) := \nabla_X^g Y - \nabla_X^\perp Y. \tag{1.80}$$

Definition Let γ be a geodesic on (M,g). A vector field $J \in \mathfrak{X}_\gamma(M)$ on γ is an *Jacobi field* if the *Jacobi equation*

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}J = R^g(J,\dot\gamma)\dot\gamma \tag{1.81}$$

is satisfied. Provided γ is normal to N, a vector field $J \in \mathfrak{X}_\gamma(M)$ on γ is a *N-Jacobi field* if

(i) $J(0) \in T_{\gamma(0)}N$
(ii) $\pi^N\left(\nabla_{\dot\gamma}J\right)(0) = \tilde{II}(J(0),\dot\gamma(0))$
(iii) J is Jacobi field on γ.

As a matter of fact for each $X, Y \in T_pM$ there is a unique Jacobi field J on γ satisfying $J(0) = X$ and $\left(\nabla_{\dot\gamma}J\right)(0) = Y$ [O'N83, Lemma 8.5]. Furthermore in case of the geodesic γ being normal to the submanifold N then a Jacobi field J is the variation vector field of a variation of γ through normal geodesics if and only if it is an N-Jacobi field on γ [O'N83, Proposition 10.28]. A *variation* of $\gamma : I = [t_0, t_1] \to M$ is a family of curves

$$\delta\gamma : I \times (-\epsilon, \epsilon) \to M \tag{1.82}$$

such that $\forall t \in I : \gamma(t) = \delta\gamma(t,0)$. It is a *geodesic variation*, if $\delta\gamma(\cdot,s)$ are geodesics for all fixed $s \in (-\epsilon,\epsilon)$. If the derivative with respect to the second parameter is indicated by an prime then the vector field $\delta\gamma'(t,0)$ on γ is the *variation vector field* of $\delta\gamma$.

Definition Let γ be a geodesic normal to N. A point $\gamma(t)$ with $t \neq 0$ is a *focal point* of N along γ if there exists a non-vanishing N-Jacobi field J on γ with $J(t) = 0$. In case where $N = \{p\}$ is a point, a focal point is called *conjugate point* along γ with (i) and (ii) in the definition of N-Jacobi fields being replaced by $J(0) = 0$.

Let $\gamma : [0, \alpha)$ be a geodesic normal to N. Then $\gamma(t)$ is a focal point of N along γ if and only if the normal exponential map $\exp : TN^\perp \to M$ is singular at $t\dot{\gamma}(0)$, i.e. $\ker [d\exp]_{t\dot{\gamma}(0)} \neq \{0\}$ [O'N83, Proposition 10.30]. In case where $N = \{p\}$ is a point, the normal exponential map is replaced by the exponential map $\exp_p$ in p [O'N83, Proposition 10.10]. Consequently there are no conjugate points of p in $\mathcal{U}$ along radial geodesics in $\mathcal{U}$ if $\mathcal{U}$ is a normal neighbourhood of p.

An application of Jacobi fields is the Causality Theorem, which roughly states that any causal curve starting on N can be approximated by a timelike curve unless the curve is a null geodesic without focal points.

Theorem 1.2.7. *(Causality Theorem [O'N83, Theorem 10.51]) Let N be a spacelike submanifold in (M, g) and $\gamma : I = (t_0, t_1) \to M$ a causal, piecewise smooth curve with $\gamma(t_0) \in N$, $\gamma(t_1) =: x$. Then there is a timelike, piecewise smooth curve arbitrarily near γ unless γ is a null geodesic normal to N, without focal point of N before x.*

Important to this thesis will be the property of causal structures, defined by normal neighbourhoods.

Proposition 1.2.8. *Let $\mathcal{U}$ be a normal neighbourhood of $p \in M$ and $\mathfrak{U} := \exp_p^{-1}(\mathcal{U})$. Then one has*

$$I^+(p,\mathcal{U}) = \exp_p\left(\mathfrak{T}_p^\uparrow \cap \mathfrak{U}\right) \qquad I^-(p,\mathcal{U}) = \exp_p\left(\mathfrak{T}_p^\downarrow \cap \mathfrak{U}\right) \tag{1.83}$$

$$J^+(p,\mathcal{U}) = \exp_p\left(\mathfrak{K}_p^\uparrow \cap \mathfrak{U}\right) \qquad J^-(p,\mathcal{U}) = \exp_p\left(\mathfrak{K}_p^\downarrow \cap \mathfrak{U}\right) \tag{1.84}$$

$$\mathcal{C}_p^+(\mathcal{U}) = \exp_p\left(\mathfrak{C}_p^\uparrow \cap \mathfrak{U}\right) \qquad \mathcal{C}_p^-(\mathcal{U}) = \exp_p\left(\mathfrak{C}_p^\downarrow \cap \mathfrak{U}\right) \tag{1.85}$$

and moreover

$$\mathcal{C}_p^+(\mathcal{U}) = E^+(p,\mathcal{U}) \qquad \mathcal{C}_p^-(\mathcal{U}) = E^-(p,\mathcal{U}). \tag{1.86}$$

A proof for $I^+(p,\mathcal{U})$ is given in the appendix. The remaining equalities for I^-, J^+, J^-, $\mathcal{C}^+$ and $\mathcal{C}^-$ are analogously proven. The last claim now is a consequence of $\exp_p$ being a bijective map on $\mathcal{U}$ and hence

$$\begin{aligned} E^+(p,\mathcal{U}) &= J^+(p,\mathcal{U}) \setminus I^+(p,\mathcal{U}) \\ &= \exp_p\left(\left(\mathfrak{K}_p^\uparrow \setminus \mathfrak{T}_p^\uparrow\right) \cap \mathfrak{U}\right) \\ &= \exp_p\left(\mathfrak{C}_p^\uparrow \cap \mathfrak{U}\right). \end{aligned}$$

Remembering the fact that the Causality Theorem 1.2.7 states that there is a timelike curve arbitrary near to a causal curve γ, which connects two points p and q in M, unless γ is a null geodesic without conjugate points, has an important consequence. If $q \in E^+(p, T)$ is a point in the future horismos of p relative to some set $T \subset M$, then any future-directed causal curve that connects p and q is a null geodesic without conjugate points.

1.3 THE MATRIX LIE GROUP $SO(n)$

The next section will summarise useful facts on the matrix Lie group $SO(n)$, i.e. the set of orthogonal matrices with determinant 1. Usually its property of being a Lie group with Lie algebra $\underline{\mathfrak{so}}(n) = T_{\mathbb{1}} SO(n)$ is spotlighted. The survey will focus on the property of $SO(n)$ to be a Riemannian manifold. In particular the exponential map and geodesics are given on arbitrary points in $SO(n)$. The main intention of this section is to motivate the use of objects such as convex neighbourhood in the context of matrix Lie groups.

General Background

Let G be an arbitrary connected Lie Group of finite dimension, $\underline{\mathfrak{g}} = T_{\mathbb{1}}G$ its Lie algebra and $\exp : \underline{\mathfrak{g}} \to G$ the usual exponential map. The left translation $L_A : G \ni B \mapsto A \cdot B \in G$ is on the one hand used to uniquely identify a left invariant vector field $X \in \mathfrak{X}(G)$ with its value at one point as $X(A) = d\left(L_{AB^{-1}}\right)_B (X(B))$.. So in particular the left translation identifies the tangent spaces of G in different points by observing $T_A G = dL_A\underline{\mathfrak{g}}$. On the other hand it provides a more general expression of the exponential map $\exp_A : T_A G \to G$ for each point in G

$$\exp_A := L_A \circ \exp \circ d\left(L_{A^{-1}}\right)_A .$$

Now consider a scalar product $\langle \cdot, \cdot \rangle$ on $\underline{\mathfrak{g}}$. This uniquely corresponds to a left invariant metric g on G. For vector fields $X, Y \in \mathfrak{X}(G)$ this correspondence is provided by $g_A\left(X_A, Y_A\right) = \langle \omega(X)_A, \omega(Y)_A \rangle$, where ω is the Maurer-Cartan form, i.e. $\omega(X)_A = d\left(L_{A^{-1}}\right)_A (X_A) \in \underline{\mathfrak{g}}$. Moreover the metric g on G is bi-invariant if and only if the corresponding scalar product is Ad-invariant. In the latter case one in addition has the equality of the exponential map $\exp_A$ defined by the flow of left invariant vector fields and the exponential map, defined by the Levi-Civita connection of g.

Now consider G left-acting transitively on a manifold M. As a consequence of transitivity the map $\underline{\mathfrak{g}} \ni X \mapsto \widetilde{X}_x \in T_x M$, which maps elements of the Lie algebra to the value of the corresponding fundamental vector field $\widetilde{X}_x = \left.\frac{d}{dt}\right|_{t=0} (\exp(-tX) \cdot x)$ at x is surjective. As the left action $l_A : M \ni x \mapsto Ax \in M$ is a diffeomorphism, one also has $d\left(l_A\right)_x (T_x M) = T_{Ax}M$. The *stabiliser* subgroup $\mathrm{stab}(x) \subset G$ of $x \in M$ is defined by $\{A \in G | A \cdot x = x\}$. It is a Lie subgroup of G and its connected component containing the identity is generated by the sub Lie algebra

$$\underline{\mathfrak{stab}}(x) := \left\{X \in \underline{\mathfrak{g}} \,|\, \exp(tX) \cdot x = x \text{ for all } t \in \mathbb{R}\right\}$$

As $\exp(0) \cdot x = x$, the last equation may equivalently be written as

$$\underline{\mathfrak{stab}}(x) = \left\{X \in \underline{\mathfrak{g}} \,|\, \widetilde{X}_x = 0\right\}.$$

The Special Case of $SO(n)$

We will now turn to the case where $M = S^{n-1}$ is a submanifold of $\mathbb{R}^n$ and G is the matrix Lie Group $SO(N)$ acting transitively on S^{n-1} by left multiplication. The Killing form $\langle h, h' \rangle = \operatorname{tr}\left(h \cdot h'^t\right)$ gives an Ad-invariant scalar product on $\underline{\mathfrak{so}}(n)$ and hence defines a bi-invariant metric g on $SO(n)$. This guarantees that the usual matrix exponential map $\exp : \underline{\mathfrak{so}}(n) \to SO(n)$ can be used to calculate the exponential map $\exp^g$, which arises from the Levi-Civita connection associated to that metric. The general theory then immediately provides the following facts

$$\begin{aligned} T_A SO(n) &= A \cdot \underline{\mathfrak{so}}(n) && (1.87)\\ T_x S^{n-1} &= \underline{\mathfrak{so}}(n) \cdot x \\ T_A SO(n) \cdot x &= T_{A \cdot x} S^{n-1} && (1.88)\\ \underline{\mathfrak{stab}}(x) &= \{X \in \underline{\mathfrak{so}}(n) \,|\, X \cdot x = 0\} && (1.89)\\ \exp^g_A(h) &= A \cdot \exp\left(A^{-1}h\right). \end{aligned}$$

Corollary 1.3.1. *Let $\mathcal{U}$ be a convex neighbourhood of some point in $SO(n)$ and $A, B \in \mathcal{U}$, then there is a unique $\tilde{h} \in \left(\exp^g_A\right)^{-1}(\mathcal{U}) \subset T_A SO(n)$ and hence a unique $h := A^{-1}\tilde{h} \in \underline{\mathfrak{so}}(n)$ such that*

$$\begin{array}{rcc} \gamma : \ [0,1] & \to & \mathcal{U} \subset SO(n) \\ t & \mapsto & \exp^g_A(t\tilde{h}) = A \cdot \exp(t \cdot h) \end{array} \qquad (1.90)$$

defines a geodesic within $\mathcal{U}$ starting at $\gamma(0) = A$ and ending at $\gamma(1) = B$.

Definition 1.3.2. A convex neighbourhood $\mathcal{U}(\mathbb{1}) \subset SO(n)$ of $\mathbb{1}$ will be called a *very convex* neighbourhood if there is a larger normal neighbourhood $\tilde{\mathcal{U}}(\mathbb{1}) \supset \mathcal{U}(\mathbb{1})$ such that for all $A, B \in \mathcal{U}(\mathbb{1})$, the product $A^{-1} \cdot B$ is in $\tilde{\mathcal{U}}(\mathbb{1})$ and if it is small enough, such that the equivalence $(\exp(X) \cdot x = x \Leftrightarrow X \cdot x = 0)$ holds for all $\exp(X) \in \tilde{\mathcal{U}}(\mathbb{1})$ and $x \in S^{n-1}$.

There really exists a very convex neighbourhood of $\mathbb{1}$. First one observes that the existence proof for convex neighbourhoods [O'N83] implies that the preimage of any ball B_δ in normal coordinates is a convex set, if δ has been chosen small enough. This preimage will act as $\tilde{\mathcal{U}}(\mathbb{1})$. A further shrinking of the ball then ensures that the equivalence $(\exp(X) \cdot x = x \Leftrightarrow X \cdot x = 0)$ holds on $\tilde{\mathcal{U}}(\mathbb{1})$. Continuity of the multiplication then provides the smaller neighbourhood $\mathcal{U}(\mathbb{1})$, such that $\mathcal{U}(\mathbb{1}) \cdot \mathcal{U}(\mathbb{1}) \subset \tilde{\mathcal{U}}(\mathbb{1})$, where $\mathcal{U}(\mathbb{1}) \cdot \mathcal{U}(\mathbb{1}) = \{A \cdot B | A, B \in U(1)\}$.

Lemma 1.3.3. *Let $\mathcal{U}$ be a very convex neighbourhood of $\mathbb{1} \in SO(n)$ and $\tilde{\mathcal{U}}$ the normal neighbourhood of $\mathbb{1}$, such that $\mathcal{U}^{-1} \cdot \mathcal{U} \subset \tilde{\mathcal{U}}$. Consider $A, B \in \mathcal{U}$ and let $h \in \underline{\mathfrak{so}}(n)$ be defined by $\exp(h) = A^{-1} \cdot B \in \tilde{\mathcal{U}}$. Then the unique geodesic from A to B is given by*

$$\begin{array}{rccc} \gamma: & [0,1] & \to & \mathcal{U} \\ & t & \mapsto & A \cdot \exp(th) \,. \end{array}$$

Proof: Obviously γ connects the two points, since $\gamma(0) = A$ and $\gamma(1) = B$. In principle γ could leave $\mathcal{U}$. So the first thing to do is to show that $\gamma(t) \in \mathcal{U}$ for all $t \in [0,1]$. The above map can be rewritten to

$$\begin{aligned} \gamma(t) &= A \cdot \exp\left(A^{-1} \cdot tAh\right) \\ &= \exp_A^g(tAh) \,. \end{aligned}$$

Ah is an element of $T_A SO(n)$. It remains to show that it is in $\left(\exp_A^g\right)^{-1}(\mathcal{U})$. Since $\mathcal{U}$ is a convex neighbourhood, by Corollary 1.3.1 there is a unique $\tilde{h} \in \left(\exp_A^g\right)^{-1}(\mathcal{U})$ such that the curve $\tilde{\gamma}$ with $\tilde{\gamma}(t) := \exp_A^g(t\tilde{h}) \in \mathcal{U}$ is a geodesic from A to B. Left multiplication with A^{-1} then maps this geodesic to a geodesic $L_{A^{-1}}\tilde{\gamma} : [0,1] \to \tilde{\mathcal{U}}$ from $\mathbb{1}$ to $A^{-1}B$ completely within $\tilde{\mathcal{U}}$. Using $\mathcal{U}$ to be a normal neighbourhood the translated geodesic $A^{-1}\gamma$ is the unique geodesic from $\mathbb{1}$ to $A^{-1}B$. The curve defined by $A^{-1}\gamma(t)$ also is a geodesic from $\mathbb{1}$ to $A^{-1}B$. With $h \in \exp^{-1}(\tilde{\mathcal{U}})$ it is completely within $\tilde{\mathcal{U}}$ and hence γ and $\tilde{\gamma}$ must coincide. Consequently $Ah = \tilde{h} \in \left(\exp_A^g\right)^{-1}(\mathcal{U})$ and hence $\gamma(t) \in \mathcal{U}$ for all $t \in [0,1]$ by convexity of $\mathcal{U}$. ■

Lemma 1.3.4. *Let $\mathcal{U}$ be a very convex neighbourhood of the identity, $\mathfrak{U} := \exp^{-1}(\mathcal{U}) \subset \underline{\mathfrak{so}}(n)$ and $x \in S^{n-1} \subset \mathbb{R}^n$. Let $h_1, h_2 \in \mathfrak{U}$ such that $\exp(h_1) \cdot x = \exp(h_2) \cdot x =: y \in S^{n-1}$. then it holds*

(i) $\exp(-h_1) \cdot \exp(h_2) \in \mathrm{stab}(x)$

(ii) *There is a map $\eta : [0,1] \to \mathfrak{U}$, such that $\eta(0) = h_1$, $\eta(1) = h_2$ and $\exp(\eta(t)) \cdot x = y$ for all $t \in [0,1]$*

Proof: The first point is due to the initial assumption on h_1 and h_2, namely $\exp(h_1) \cdot x = \exp(h_2) \cdot x$. Now $\mathcal{U}$ is very convex, such that in particular $\exp(-h_1)\exp(h_2) =: \exp(h)$ is an element of $\tilde{\mathcal{U}}$ and (i) is equivalent to the requirement $h \cdot x = 0$. By Lemma 1.3.3 the curve $\gamma(t) = \exp(h_1) \cdot \exp(t \cdot h)$ is a geodesic in $\mathcal{U}$ connecting $\exp(h_1) = \gamma(0)$ and $\exp(h_2) = \gamma(1)$. By using $h \cdot x = 0$ and substituting it into the expansion of exp one then obtains

$$\begin{aligned} \gamma(t) \cdot x &= \exp(h_1) \cdot \exp(th) \cdot x \\ &= \exp(h_1) \cdot (th)^0 \cdot x \\ &= \exp(h_1) \cdot \mathbb{1} \cdot x \\ &= y. \end{aligned}$$

Defining $\eta(t) := \exp^{-1}(\gamma(t))$ proves the second claim. ■

1.4 CONFORMALLY EINSTEIN MANIFOLDS

This section will provide basic definitions of Einstein manifolds and conformal geometry. First the different concepts of Einstein metrics will be discussed. Next conformal transformation rules will be given and finally almost Einstein structures — a conformal analogue of Einstein manifolds — will be introduced.

1.4.1. Einstein Manifolds

In the following the focus will be on manifolds $(M^{n>2}, g)$, for which the Ricci tensor is a multiple of the metric or which are at least conformal to such manifolds. Curvature quantities that depend on the metric in this section will refer to the metric g, the index g then is dropped, e.g. $\tau = \tau^g$.

Definition 1.4.1. A n-dimensional semi-Riemannian manifold (M, g) is said to be *Einstein* , if the trace-free part of the Ricci tensor

$$E[g] := \mathrm{Ric} - \frac{\tau}{n} g, \tag{1.91}$$

vanishes, i.e. $E[g] = 0$. $E[g]$ will be referred to as *Einstein tensor* if the metric is fixed or as Einstein operator, if interpreted as operator $E : \mathrm{Met}(M) \to \mathcal{S}^2 M$ on metrics of M with values in the symmetric (2,0) tensors on M.

In physics literature the name Einstein tensor refers to the divergence-free tensor

$$G[g] := \mathrm{Ric} - \frac{\tau}{2} g. \tag{1.92}$$

The two tensors $E[g]$ and $G[g]$ coincide on manifolds with vanishing scalar curvature τ. Manifolds (M, g) with Lorentzian metric g and $G[g] = 0$ are called *vacuum space-times*. In dimension $n > 2$ this implies vanishing of the scalar curvature. A treatment that meld those two Einstein tensors involves what physicists call a *cosmological term*. A metric g is said to fulfil the vacuum Einstein equation with *cosmological term* Λg if

$$G[g] + \Lambda g = 0. \tag{1.93}$$

By taking the trace one immediately gets $\tau = \frac{2n}{n-2}\Lambda$ and taking the divergence gives constancy of Λ and hence of τ. In particular g fulfils a vacuum Einstein equation with cosmological term if and only if E[g] = 0. By choosing Λ one prescribes the scalar curvature of the corresponding Einstein metric. So by choosing $\Lambda = \pm\frac{(n-2)n}{2}$ one gets the equivalent normalised equation

$$\mathrm{Ric} = \pm n g. \tag{1.94}$$

It is normalised in the sense that it implies $\tau = n^2$ on solutions. Another frequently-used choice is $\Lambda = \pm\frac{(n-2)(n-1)}{2}$, which is equivalent to $\mathrm{Ric} = \pm(n-1)g$ and implies $\tau = n(n-1)$.

The condition of being Einstein can be rewritten in terms of the Schouten tensor. For that one observes that the trace-free part of P

$$\mathrm{P}_0 = \mathrm{P} - \frac{1}{n} J g = \frac{1}{n-2}\left(\mathrm{Ric} - \frac{\tau}{n} g\right) \tag{1.95}$$

is the Einstein tensor up to a constant. Vanishing of P_0 is equivalent for the metric to be an Einstein metric. A necessary criterion of a metric for being Einstein appears as follows. Let (M, g) be an pseudo-Riemannian Einstein manifold of dimension $n > 3$. Then the Schouten tensor is a constant multiple of the metric, $\mathrm{P} = \frac{1}{2n(n-2)}\tau g$, with constant factor $\frac{1}{2n(n-2)}\tau$. Consequently its covariant derivative $\nabla^g \mathrm{P}$ vanishes such that $\mathrm{C}(X, Y, Z) \overset{(1.40)}{=} \left(\nabla^g_Y \mathrm{P}\right)(Z, X) -$

$\left(\nabla^g_Z \mathrm{P}\right)(Y, X) = 0$. Vanishing of the Cotton tensor on the other hand is equivalent to vanishing divergence of the Weyl tensor by Equation (1.39). Hence in dimension $n > 3$ a metric can be Einstein only if

$$\operatorname{div} \mathrm{W} = 0. \tag{1.96}$$

Closely related to manifolds that admit an Einstein metric is the following construction. It is a result of tractor calculus [BEG94].

Definition 1.4.2. Let $(M^{p,q}, g)$ be a pseudo-Riemannian manifold and $L(M)$ its trivial real *line bundle*. The set of smooth sections $\Gamma(L(M))$ is isomorphic to the space of smooth maps $C^\infty(M)$ on M. Further let ∇ be the Levi-Civita connection. Then the vector bundle $(\mathcal{T}^g, \mathbf{g}, \nabla^{\mathcal{T}^g})$ over M with connection $\nabla^{\mathcal{T}^g}$ and metric $\mathbf{g}$ is defined by the following requirements. The bundle is

$$\mathcal{T}^g := L(M) \oplus TM \oplus L(M). \tag{1.97}$$

Consider sections $(f_i, X_i, h_i) \in \Gamma(\mathcal{T}^g)$, then the metric is

$$\mathbf{g}\left((f_1, X_1, h_1), (f_2, X_2, h_2)\right) := f_1 h_2 + f_2 h_1 + g(X_1, X_2). \tag{1.98}$$

The connection $\nabla^{\mathcal{T}^g} : \Gamma(\mathcal{T}) \to \Gamma(T^*M \otimes \mathcal{T}^g)$ is provided by

$$\nabla^{\mathcal{T}^g}_Y \begin{pmatrix} f \\ X \\ h \end{pmatrix} := \begin{pmatrix} Y(f) - g(X, Y) \\ \nabla_Y X + hY + f\mathrm{P}^\sharp(Y) \\ Y(h) - \mathrm{P}(X, Y) \end{pmatrix}. \tag{1.99}$$

From the definition one immediately has that $\mathbf{g}$ is of index $(p+1, q+1)$ and non-degenerate. Furthermore by generically extending $\nabla^{\mathcal{T}^g}$ to a connection on tensor products of $\mathcal{T}^{g*}$ in the sense of Equation (1.1) one gains compatibility of $\nabla^{\mathcal{T}^g}$ and $\mathbf{g}$. This can easily be seen by considering sections $\theta_i = (f_i, X_i, h_i) \in \Gamma(\mathcal{T})$ and calculating

$$\begin{aligned} \mathbf{g}\left(\nabla^{\mathcal{T}^g}_Y \theta_i, \theta_j\right) = {} & \big(Y(f_i) - g(Y, X_i)\big)h_j + \big(Y(h_i) - \mathrm{P}(Y, X_i)\big)f_j \\ & + g(\nabla_Y X_i, X_j) + h_i g(Y, X_j) + f_i \mathrm{P}(Y, X_j). \end{aligned}$$

Hence

$$\begin{aligned} \mathbf{g}\left(\nabla^{\mathcal{T}^g}_Y \theta_1, \theta_2\right) + \mathbf{g}\left(\theta_1, \nabla^{\mathcal{T}^g}_Y \theta_2\right) &= h_2 Y(f_1) + f_2 Y(h_1) + h_1 Y(f_2) + f_1 Y(h_2) \\ &\quad + g(\nabla_Y X_1, X_2) + g(\nabla_Y X_2, X_1) \\ &= Y\big(f_1 h_2 + f_2 h_1 + g(X_1, X_2)\big) \\ &= Y\big(\mathbf{g}(\theta_1, \theta_2)\big). \end{aligned}$$

and consequently one obtains $\left(\nabla^{\mathcal{T}^g}_Y \mathbf{g}\right)(\theta_1, \theta_2) = 0$.

1.4.2. Conformal Transformations

The initial point to start from is that of a conformal change of the metric. Let M be a smooth manifold. Two semi-Riemannian metrics g and $\tilde{g}$ will be said to be *conformally equivalent*, if there is a function $\phi \in C^\infty(M)$ such that $\tilde{g} := e^{2\phi} g$. An equivalent way is to demand that there is a smooth, strictly positive function $\omega \in C^\infty(M, \mathbb{R}^+)$ such that $\tilde{g} = \omega^2 g$. The functions $e^{2\phi}$ and ω^2 are referred to as *conformal factor* and $\tilde{g}$ is referred to as *conformal change* of g. The set of metrics that are conformally equivalent to g is the *conformal class* of g and will be denoted $[g]$.

Quantities that depend on the metric, such as the Levi-Civita connection and curvature tensors, will change under a conformal transformations of the underlying metric. The difference between such objects will be given in the next paragraphs. Consider $O[g]$ to be a tensor or an operator that depends on the pseudo-Riemannian metric g. That conformally changed quantity $O[\tilde{g}]$ will also be denoted $\tilde{O}$. A tensor $T[g]$ is said to be *conformally covariant* of weight α, if

$$T[\omega^2 g] = \omega^\alpha T[g]. \tag{1.100}$$

Lemma 1.4.3. *Let (M, g) be a semi-Riemannian manifold of dimension n. Consider a conformal change $\tilde{g} = \sigma^{-2} g$ of the metric. The following transformation laws holds for the Levi-Civita connection, Hessian of a smooth map f, Ricci curvature, Schouten tensor, scalar curvature, Weyl curvature, Schouten tensor and Bach tensor*

$$\tilde{\nabla}_X Y = \nabla_X Y - \sigma^{-1}\Big(X(\sigma)Y + Y(\sigma)X - g(X,Y)\,\mathrm{grad}\,\sigma\Big) \tag{1.101}$$

$$\tilde{\nabla}_X \omega = \nabla_X \omega - \sigma^{-1}\Big(\omega(\mathrm{grad}\,\sigma)X^\flat - X(\sigma)\omega - \omega(X)d\sigma\Big) \tag{1.102}$$

$$\begin{aligned}\widetilde{\mathrm{Hess}} f(X,Y) = {} & \mathrm{Hess}\, f(X,Y) \\ & + \sigma^{-1}\Big(df(X)d\sigma(Y) + df(Y)d\sigma(X) - d\sigma(\mathrm{grad}\, f)g(X,Y)\Big)\end{aligned} \tag{1.103}$$

$$\begin{aligned}\widetilde{\mathrm{Ric}}(X,Y) = {} & \mathrm{Ric}(X,Y) + (n-2)\sigma^{-1}\,\mathrm{Hess}\,\sigma(X,Y) - \sigma^{-1}\Delta\sigma g(X,Y) \\ & - (n-1)\sigma^{-2}\|\,\mathrm{grad}\,\sigma\|_g^2 g(X,Y)\end{aligned} \tag{1.104}$$

$$\tilde{\mathrm{P}}(X,Y) = \mathrm{P}(X,Y) + \sigma^{-1}\,\mathrm{Hess}\,\sigma(X,Y) - \frac{1}{2}\sigma^{-2}\|\,\mathrm{grad}\,\sigma\|_g^2 g(X,Y) \tag{1.105}$$

$$\tilde{\tau} = \sigma^2\tau - n(n-1)\|\,\mathrm{grad}\,\sigma\|^2 - 2(n-1)\sigma\Delta\sigma \tag{1.106}$$

$$\tilde{\mathrm{W}} = \sigma^{-2}\,\mathrm{W} \tag{1.107}$$

$$\tilde{\mathrm{C}} = \mathrm{C} - \sigma^{-1}\,\mathrm{grad}\,\sigma \lrcorner \mathrm{W} \tag{1.108}$$

$$\begin{aligned}\tilde{\mathfrak{B}}(X,Y) = {} & \sigma^{-2}\Big(\mathfrak{B}(X,Y) \\ & + \sigma^{-1}(n-4)\big(\mathrm{C}(X,\mathrm{grad}\,\sigma,Y) + \mathrm{C}(Y,\mathrm{grad}\,\sigma,X)\big) \\ & + \sigma^{-2}(n-4)\,\mathrm{W}(\mathrm{grad}\,\sigma, X, \mathrm{grad}\,\sigma, Y)\Big)\,.\end{aligned} \tag{1.109}$$

where all objects on the right-hand side are taken with respect to the unchanged metric g.

The conformal transformation rule for the Levi-Civita connection are calculated in the appendix as an example. The remaining equations are then an application of those two transformations. More detailed calculations may be for example found in [Juh09].

Corollary 1.4.4. *Let (M, g) be a semi-Riemannian manifold of dimension $n \geq 4$. Then under a conformal change $\tilde{g} = \sigma^{-2} g$, the divergence of the Weyl tensor transforms as*

$$\sigma^{3-n}\widetilde{\mathrm{div}}\tilde{\mathrm{W}} = \mathrm{div}\,\sigma^{3-n}\,\mathrm{W}\,. \tag{1.110}$$

This gives a necessary condition to a metric g, to be conformally related to an Einstein metric, namely

$$\mathrm{div}\,\sigma^{3-n}\,\mathrm{W} = 0. \tag{1.111}$$

Proof: For the divergence of the Weyl tensor one has

$$\begin{aligned}\widetilde{\mathrm{div}}\tilde{\mathrm{W}} & \overset{(1.39)}{=} -(n-3)\tilde{\mathrm{C}} \\ & \overset{(1.108)}{=} -(n-3)\Big(\mathrm{C} - \sigma^{-1}\,\mathrm{grad}\,\sigma\lrcorner\mathrm{W}\Big) \\ & = \mathrm{div}\,\mathrm{W} + (n-3)\sigma^{-1}\,\mathrm{grad}\,\sigma\lrcorner\mathrm{W}\,.\end{aligned}$$

On the other hand $\mathrm{div}\,\sigma^k\,\mathrm{W} = \sigma^k\left(\mathrm{div}\,\mathrm{W} - k\sigma^{-1}\,\mathrm{grad}\,\sigma\lrcorner\mathrm{W}\right)$ such that one can substitute $\mathrm{div}\,\mathrm{W} = \sigma^{-k}\,\mathrm{div}\,\sigma^k\,\mathrm{W} + k\sigma^{-1}\,\mathrm{grad}\,\sigma\lrcorner\mathrm{W}$ in the last equation

$$\widetilde{\mathrm{div}}\tilde{\mathrm{W}} = \sigma^{-k}\,\mathrm{div}\,\sigma^k\,\mathrm{W} + (n-3+k)\sigma^{-1}\,\mathrm{grad}\,\sigma\lrcorner\mathrm{W}\,.$$

The choice $k = 3 - n$ then gives the claim. ■

A frequently used conformal invariance is that of null geodesics. Consider γ to be a g-geodesic. By using Equation (1.101) on finds

$$\begin{aligned}\tilde{\nabla}_{\dot{\gamma}}\dot{\gamma} & = \nabla_{\dot{\gamma}}\dot{\gamma} - \sigma^{-1}\left(2d\sigma(\dot{\gamma})\dot{\gamma} - g(\dot{\gamma},\dot{\gamma})\,\mathrm{grad}\,\sigma\right) \\ & = -2\sigma^{-1}d\sigma(\dot{\gamma})\dot{\gamma}.\end{aligned}$$

Hence γ is a $\tilde{g}$-pregeodesic. This will be summarised in a lemma.

Lemma 1.4.5. *Let $\gamma : I \to M$ be a null geodesic of (M, g). Then it can be reparametrised to a geodesic of $(M, \tilde{g} = \sigma^{-2} g)$.*

1.4.3. Conformal Boundaries

Definition 1.4.6. Let $(\tilde{M}, \tilde{g})$ be a semi-Riemannian manifold of dimension n. Consider $\imath : \tilde{M}^n \hookrightarrow M^n$ to be an embedding such that the topological boundary $\partial\left(\imath(\tilde{M})\right)$ is non-empty. For simplicity it will be denoted $\partial\tilde{M}$ in the following. A *defining function* or *boundary defining function* is a smooth map $\sigma : M \to \mathbb{R}$ such that

(i) $\partial\tilde{M} \subset \sigma^{-1}(0)$

(ii) if $d\sigma_x = 0$ and $x \in \sigma^{-1}(0)$ then there is a neighbourhood U of x with $d\sigma_y \neq 0$ for all $y \in U \setminus \{x\}$.

$(\tilde{M}, \tilde{g})$ is said to be *conformally completed* if the conformally equivalent metric

$$g := \sigma^2 \imath_* \tilde{g},$$

with $(\imath_*\tilde{g})_{\imath(x)}(V, W) := \tilde{g}_x\left((d\imath_x)^{-1}(V), (d\imath_x)^{-1}(W)\right)$ for $V, W \in d\imath_x\left(T_x\tilde{M}\right)$, extends to a sufficiently smooth metric on M. The topological boundary $\partial\tilde{M}$ then is the *conformal boundary* of $\tilde{M}$ and $\overline{\imath\left(\tilde{M}\right)}$ is its *conformal completion* (in M). If the conformal completion is compact, it is called *conformal compactification* . A defining function on (M, g) is said to be a *geodesic defining function*, provided $\|\operatorname{grad}\sigma\|_g^2$ is constant on M or at least in a neighbourhood of $\partial\tilde{M}$.

If σ is a geodesic defining function, then constancy of $g(\operatorname{grad}\sigma, \operatorname{grad}\sigma)$ implies vanishing of $\nabla_{\operatorname{grad}\sigma}\operatorname{grad}\sigma$ on M and consequently the integral curves of $\operatorname{grad}\sigma$ are geodesics with respect to g (see Lemma A.1.3 for a short proof). This motivates the name geodesic defining function for such maps. In case of Riemannian signature, σ is a distant function in a neighbourhood of the conformal boundary, i.e. $\sigma(x) \propto \mathit{dist}^g\left(x, \partial\tilde{M}\right)$.

1.4.4. Conformal Density Bundles

A conformal structure $(M, [g])$ can also be treated as principal subbundle $(\mathcal{C}, \pi, M, \mathbb{R}^+)$ of the bundle of symmetric (2,0)-tensors S^2T^*M. For fixed metric g, the fibres are $\mathcal{C}_x = \{\tilde{g}_x \mid \tilde{g} \in [g]\}$. This bundle has structure group $\mathbb{R}^+$. Given the representation $\rho_\omega : \mathbb{R}^+ \ni x \mapsto (y \mapsto x^{-\frac{\omega}{2}} \cdot y) \in \operatorname{End}(\mathbb{R})$ the associated vector bundle

$$\mathcal{D}[\omega] := \mathcal{C} \times_{\rho_\omega} \mathbb{R} \tag{1.112}$$

is called the *conformal density bundle of weight* ω . It generically can be trivialised by a choice of metric $\tilde{g} \in [g]$. The trivialisation then is provided by the mapping $[g, \kappa]_x \mapsto (x, \theta^\omega(x)\kappa_x) \in M \times \mathbb{R}$, where $g = \theta^{-2}\tilde{g}$.

Tractor Bundles

An application of conformal density bundles is the construction of tractor bundles. The introduction to tractor bundles will be in terms of the jet bundle of $\mathcal{D}[1]$. Let $\pi : F \to M$ be a smooth fibre bundle over M. The equivalence class $j^k\mathcal{S}_p$ in $p \in M$ with representative $\mathcal{S} \in \Gamma(F)$ consists of all sections $\mathcal{S}' \in \Gamma(F)$, whose derivatives up to order k coincides with that of $\mathcal{S}$ in p. The k-th jet bundle of E then is defined by $J^k(E) := \bigcap_{p\in M}\left\{j^k\mathcal{S}_p \mid \mathcal{S} \in \Gamma(E)\right\}$. It also is called k-th jet prolongation of E.

Let $(M^n, [g])$ be a pseudo-Riemannian conformal structure on M and $\mathcal{D}[1]$ be the associated density bundle of weight 1. The condition of the conformal structure to allow a smooth map σ on M such that for some $g \in [g]$, $\sigma^{-2}g$ is Einstein on the support of σ is equivalent to require that σ is solution to the conformally covariant equation

$$A[g, \sigma] = \nabla^g\nabla^g\sigma + \sigma\mathrm{P}^g + \rho g = 0,$$

whith $\rho = -\frac{1}{n}\operatorname{tr}^g\left(\nabla^g\nabla^g\sigma + \sigma\mathrm{P}^g\right)$ (see section 1.4.5 for details). The covariance is with respect to conformal rescalings of type $(g, \sigma) \to (e^{2\omega}g, e^\omega\sigma)$. So consider the conformal density bundle $\mathcal{D}[1]$ of weight 1 on $(M, [g])$. Let $\mathcal{S} \in \Gamma(\mathcal{D}[1])$ be a local section and $\mathcal{S}^g : M \to \mathbb{R}$ its trivialisation with respect to metric g in the conformal class. Then $\mathcal{S}$ is said to satisfy the above equation, if $A[g, \mathcal{S}^g] = 0$ in a trivialisation. Conformal covariance then ensures that this condition does not depend on the choice of g. The short notation of the requirement will be written $A[\mathcal{S}] = 0$. If the equation is satisfied at a point $p \in M$ it will be written $A[\mathcal{S}]_p = 0$. As a matter of fact, $A[\mathcal{S}]$ in general is a local section of $\mathcal{D}[1]$.

Solutions to the equation $A[\mathcal{S}] = 0$ may not exist. But then if one considers the second jet prolongation $J^2\mathcal{D}[1]$ the equation defines a bundle $\mathcal{T}$, which pointwise is defined by $\mathcal{T}_p = \{j^2\mathcal{S}_p \mid \mathcal{S} \in \mathcal{D}[1], A[\mathcal{S}] = 0\}$. $\mathcal{T} \subset J^2\mathcal{D}[1]$ is a *tractor bundle* of $(M, [g])$. Sections of the tractor bundle are called *tractors*. The tractor bundle of $(M^{p,q}, [g])$ admits a generic splitting in a particular choice of a metric in the conformal class and is equipped with a generic tractor metric of index $(p+1, q+1)$ and a connection that is compatible with the tractor metric. The latter metric representation of tractors corresponds to the one given in definition 1.4.2. It will be reintroduced shortly.

A solution $\mathcal{S} \in \Gamma(\mathcal{D}[1])$ with $A[\mathcal{S}] = 0$ naturally induces a section of $\mathcal{T}$, which is defined by $p \mapsto j^2\mathcal{S}_p$. The converse does not hold. By choosing a representative $g \in [g]$ in the conformal class, the tractor bundle can be identified with the splitting

$$\mathcal{T} \cong \mathcal{D}[1] \oplus TM[-1] \oplus \mathcal{D}[-1],$$

where $TM[\omega] = TM \otimes \mathcal{D}[\omega]$ [BEG94, GW12]. With respect to g, a section $\mathcal{S} \in \Gamma(\mathcal{T})$ of the tractor bundle is represented by $\mathcal{S} \stackrel{g}{=} (\sigma, Y, \rho)$. Under conformal change $g \to e^{2\omega}g$, the representation transforms as

$$(\sigma\,,\,Y, \rho) \to \left(e^{\omega}\sigma, e^{-\omega}\left(Y + \sigma\,\mathrm{grad}^g\,\omega\right)\,,\,e^{-\omega}\left(\rho - d\omega(Y) - \frac{1}{2}\sigma g(\mathrm{grad}^g\,\omega, \mathrm{grad}^g\,\omega)\right)\right).$$

The *normal tractor connection* then is represented for one choice of metric by

$$\nabla^{\mathcal{T}}_X \begin{pmatrix} \sigma \\ Y \\ \rho \end{pmatrix} \stackrel{g}{=} \begin{pmatrix} d\sigma(X) - g(X, Y) \\ \nabla^g_X Y + \rho X + \sigma\left(\mathrm{P}^g\right)^\sharp(X) \\ d\rho(X) - \mathrm{P}^g(X, Y) \end{pmatrix}.$$

This may also be taken as a definition as the above equation is conformally covariant with respect to the identification given before. The tractor connection is compatible with the *tractor metric* defined in a particular choice of metric g by

$$h((\sigma_1, Y_1, \rho_1), (\sigma_2, Y_2, \rho_2)) = \sigma_1\rho_2 + \sigma_2\rho_1 + g(Y_1, Y_2).$$

This construction of the standard tractor bundle on a conformal manifold is unique up to isomorphism [ČG03], which provides equivalence to other constructions found in literature [BJ10, GW12]. Important to this introduction is the fact that the ambient metric construction by Fefferman and Graham provides a realisation of the tractor bundle, where the tractor metric and normal tractor connection appear as a restriction of the ambient metric and its Levi-Civita connection [GW12].

1.4.5. Almost Einstein Manifolds

The subject of diverse considerations in this thesis are those metrics that are conformal to an Einstein metric, i.e. there is a conformally rescaled metric $\tilde{g} = \sigma^{-2}g$ such that

$$\widetilde{\mathrm{Ric}} = \frac{\tilde{\tau}}{n}\tilde{g},$$

for a non-vanishing function σ. A slight generalisation is gained, if σ is allowed to vanish on a Lebesgue null set. Then g will only be required to be conformally Einstein away from the zero set of σ. This leads to structures that are called almost Einstein [Gov05, Gov10]. The results, which are presented in the following section, are extracts of the last two papers. Nevertheless the proofs will in difference to the papers be given without using tractor calculus or tractor indices.

Let (M, g) be an n-dimensional pseudo-Riemannian manifold and $\sigma \in C^\infty(M)$ a smooth function. The trace-free tensor

$$A[g, x] := \operatorname{Hess}^g \sigma + \sigma \mathrm{P}^g + \rho g \tag{1.113}$$

is called an *almost Einstein tensor*, where ρ is the trace

$$\begin{aligned} \rho :=& -\frac{1}{n} \operatorname{tr}^g(\operatorname{Hess}^g \sigma + \sigma \mathrm{P}) \\ =& \frac{1}{n}(\Delta^g \sigma - J\sigma) \end{aligned} \tag{1.114}$$

and $J = \frac{\tau}{2(n-1)}$ is trace of the Schouten tensor.

A generalisation of the scalar curvature is defined by

$$\begin{aligned} S[g, \sigma] :=& \frac{2}{n}\sigma(\mathrm{J} - \Delta^g)\sigma - \| \operatorname{grad} \sigma \|_g^2 \\ =& -2\sigma\rho - \| \operatorname{grad} \sigma \|_g^2 . \end{aligned} \tag{1.115}$$

It is referred to as *almost scalar curvature*.

Definition 1.4.7. A triple (M, g, σ) with (M, g) being a pseudo-Riemannian manifold and $\sigma \in C^\infty(M)$ a smooth function is said to be an *almost Einstein structure* if its almost Einstein tensor vanishes,

$$A[g, \sigma] \equiv 0 \tag{1.116}$$

and if σ only vanishes on a Lebesgue null set. A manifold M which admits an almost Einstein structure is said to be an *almost Einstein manifold*. The zero set of σ is denoted by

$$\Sigma := \sigma^{-1}(0). \tag{1.117}$$

It is also called *singularity set* of (M, g, σ).

Definition 1.4.8. The triple (M, g, σ) is said to be *almost scalar constant* if

$$S[g, \sigma] = const. \tag{1.118}$$

Lemma 1.4.9. *Let (M, g, σ) be an almost Einstein structure. Then the exterior derivative of $\rho = \frac{1}{n}(\Delta\sigma - J\sigma)$ is*

$$d\rho = \mathrm{P}^\sharp(d\sigma). \tag{1.119}$$

Proof: The almost Einstein tensor $A[g, \sigma]$ vanishes on M and so does its divergence. Using Equation (1.2) for the Hessian of σ one obtains for its divergence $\operatorname{div}(\nabla\nabla\sigma) = \nabla^*\nabla(\nabla\sigma) = \Delta^{\nabla^g}\nabla\sigma$. The Weitzenböck identity (1.14) can be applied to the last term such that

$$\begin{aligned} 0 &= \operatorname{div} A[g, \sigma] \\ &= \Delta^{\nabla^g} d\sigma + \operatorname{div}(\sigma \mathrm{P}) + \operatorname{div}(\rho g) \\ &\overset{(1.15),(1.34)}{=} \nabla(\Delta^g \sigma) - \operatorname{Ric}^\sharp(\operatorname{grad} f) - \mathrm{P}^\sharp(\operatorname{grad}\sigma) - \sigma dJ - d\rho \end{aligned} \tag{1.120}$$

Now by taking the exterior derivative of the metric trace of $A[g, \sigma]$ one obtains

$$\begin{aligned} 0 &= \nabla \operatorname{tr}^g A \\ &= \nabla(-\Delta^g \sigma + \sigma J + n\rho) \\ &= -\nabla\Delta^g\sigma + J d\sigma + \sigma dJ + n d\rho \\ &= -\nabla\Delta^g\sigma + J g^\sharp(\operatorname{grad}\sigma) + \sigma dJ + n d\rho \end{aligned} \tag{1.121}$$

The sum of (1.120) and (1.120) then gives the claim

$$\begin{aligned} 0 &= (-\operatorname{Ric} + Jg - \mathrm{P})^\sharp(\operatorname{grad}\sigma) + (n-1)d\rho \\ &= \big((2-n)\mathrm{P} - \mathrm{P}\big)^\sharp(\operatorname{grad}\sigma) + (n-1)d\rho. \end{aligned}$$

■

Corollary 1.4.10. *Let (M, g, σ) be an almost Einstein structure, then the Laplacian of ρ is*

$$\Delta^g \rho = \sigma \|\mathrm{P}\|_g^2 + \rho J - dJ(\operatorname{grad}\sigma). \tag{1.122}$$

The equation immediately emerges, if the divergence of Equation (1.119) is calculated. By calculation of the exterior derivative of $S[g,\sigma]$ for a fixed structure, one obtains another consequence.

Corollary 1.4.11. *Let $(M, g.\sigma)$ be a connected almost Einstein structure. Then it is almost scalar constant, i.e. $S[g,\sigma] = const$.*

Proof: Consider $S[g,\sigma] = -2\rho\sigma - \|\operatorname{grad}\sigma\|_g^2$. To show constancy of $S[g,\sigma]$ it suffices to show $dS = 0$

$$\begin{aligned} dS(X) &= -2(\sigma d\rho + \rho d\sigma)(X) - \nabla_X g(\operatorname{grad}\sigma, \operatorname{grad}\sigma) \\ &\overset{(1.119)}{=} -2\big(\sigma P(\operatorname{grad}\sigma, X) + \rho g(\operatorname{grad}\sigma, X) - 2g(\nabla_X \operatorname{grad}\sigma, \operatorname{grad}\sigma)\big) \\ &= -2A[g,\sigma](\operatorname{grad}\sigma, X). \end{aligned}$$

Since the almost Einstein tensor vanishes on almost Einstein structures, so does the differential of S and therefore S is constant on M. ■

The motivation for calling the above defined structure almost Einstein arises from the property of g to be conformal to an Einstein metric on a dense subset of M.

Remark. Let (M, g) be a pseudo-Riemannian manifold of dimension n and σ a smooth function, whose singularity set is a Lebesgue null set. Away from the singularity the almost Einstein tensor and almost scalar curvature of (g, σ) are related to the Einstein tensor $E[\tilde{g}]$ and scalar curvature $\tilde{\tau}$ of $\tilde{g} = \sigma^{-2} g$ by

$$A[g,\sigma] = \frac{\sigma}{n-2}\left(\widetilde{\operatorname{Ric}} - \frac{\tilde{\tau}}{n}\tilde{g}\right) \tag{1.123}$$

$$S[g,\sigma] = \frac{\tilde{\tau}}{n(n-1)}. \tag{1.124}$$

So in particular if (M, g, σ) is an almost Einstein structure, then $\tilde{g}$ is an Einstein metric. The tensors $A[g,\sigma]$ and $S[g,\sigma]$ rescale under a conformal change $(g,\sigma) \to \big(e^{2\omega} g, e^{\omega}\sigma\big)$ as

$$\begin{aligned} A[e^{2\omega} g, e^{\omega}\sigma] &= e^{\omega} A[g,\sigma] \\ S[e^{2\omega} g, e^{\omega}\sigma] &= S[g,\sigma]. \end{aligned}$$

Proof: By calculating the right-hand side of Equation (1.123) away from the singularity set of σ one obtains

$$\begin{aligned} \frac{\sigma}{n-2}\left(\widetilde{\operatorname{Ric}} - \frac{\tilde{\tau}}{n}\tilde{g}\right) &\overset{(1.104)}{\underset{(1.106)}{=}} \frac{\sigma}{n-2}\Bigg(\operatorname{Ric} + \frac{n-2}{\sigma}\operatorname{Hess}^g\sigma - \frac{1}{\sigma}\Delta^g\sigma g - \frac{n-1}{\sigma^2}\|\operatorname{grad}\sigma\|^2 g \\ &\qquad - \frac{1}{n\sigma^2}\left[\sigma^2\tau - n(n-1)\|\operatorname{grad}\sigma\|^2 - 2(n-1)\sigma\Delta^g\sigma\right] g\Bigg) \\ &= \frac{\sigma}{n-2}\left(\operatorname{Ric} - \frac{1}{n}\tau + \frac{n-2}{\sigma}\left[\operatorname{Hess}^g\sigma + \frac{1}{n}\Delta^g\sigma g\right]\right) \\ &\overset{(1.95)}{=} \frac{\sigma}{n-2}\left((n-2)\mathrm{P}_0 + \frac{n-2}{\sigma}\operatorname{Hess}_0^g\sigma\right) \\ &= A[g,\sigma] \end{aligned}$$

Rewriting the conformal transformation law (Equation (1.106) for the scalar curvature in terms of J and σ gives

$$\begin{aligned}\tilde{\tau} &\overset{(1.106)}{=} \sigma^2\tau - n(n-1)\|\operatorname{grad}\sigma\|^2 - 2(n-1)\sigma\Delta\sigma \\ &= 2(n-1)\sigma^2 J - n(n-1)\|\operatorname{grad}\sigma\|^2 - 2(n-1)\sigma\Delta\sigma \\ &= n(n-1)\left[\frac{2}{n}\sigma(J-\Delta)\sigma - \|\operatorname{grad}\sigma\|^2\right] \\ &= n(n-1)S[g,\sigma],\end{aligned}$$

which provides Equation (1.124). Now denote $g' = e^{2\omega}g$ and $\sigma' = e^{\omega}\sigma$ and observe that the almost Einstein tensor may also be written as trace-free part of the Schouten tensor $\tilde{\mathrm{P}}$, i.e. $A[g,\sigma] = \sigma\tilde{\mathrm{P}}_0$ for $\tilde{g} = \sigma^{-2}g$. So in particular

$$\begin{aligned}A[e^{2\omega}g, e^{\omega}\sigma] &= A[g',\sigma'] \\ &= \sigma'\hat{\mathrm{P}}_0,\end{aligned}$$

with $\hat{g} = \sigma'^{-2}g' = \sigma^{-2}g = \tilde{g}$. Therefore $\hat{\mathrm{P}}_0 = \tilde{\mathrm{P}}_0$ and so

$$\begin{aligned}A[e^{2\omega}g, e^{\omega}\sigma] &= e^{\omega}\sigma\tilde{\mathrm{P}}_0 \\ &= e^{\omega}A[g,\sigma].\end{aligned}$$

By using Equation (1.124) and the same argument as above one then obtains

$$S[e^{2\omega}g, e^{\omega}\sigma] = \frac{1}{n(n-1)}\hat{\tau} = \frac{1}{n(n-1)}\tilde{\tau} = S[g,\sigma].$$

■

Of special interest are those almost Einstein structures (M,g,σ) whose metrics are conformally related to a Ricci-flat metric. Such structures will then be called *almost Ricci-flat*. It is equivalent to demand $S[g,\sigma] = 0$.

Along the singularity set Σ of almost Einstein structures one can conclude an additional set of properties.

Corollary 1.4.12. *Let (M,g,σ) be an almost Einstein structure, then*

(i) the metric is proportional to the Hessian of σ along Σ

$$\operatorname{Hess}^g\sigma =|_\Sigma -\rho g.$$

(ii) vanishing of $d\sigma_p$ and $\sigma(p)$ at $p \in M$ implies $\rho(p)$ does not to vanish at p.

(iii) if (M,g,σ) is an almost Ricci-flat almost Einstein structure, then $\operatorname{grad}\sigma$ is a null vector or vanishes.

Proof: Along the singularity set Σ, the almost Einstein tensor has the form $A[g,\sigma] =|_\Sigma \operatorname{Hess}^g\sigma + \rho g$, which gives the first property. On the other hand $S[g,\sigma] =|_\Sigma - \|\operatorname{grad}\sigma\|_g^2$ provides the second property. The third statement is proven by considering the bundle $\mathcal{T}$ defined in (1.97). A special section of that bundle is defined by $(\sigma, \operatorname{grad}\sigma, \rho) \in \Gamma(\mathcal{T})$. Evaluating the connection (1.99) on this section gives

$$\nabla^{\mathcal{T}}_X \begin{pmatrix}\sigma \\ \operatorname{grad}\sigma \\ \rho\end{pmatrix} = \begin{pmatrix} X(\sigma) - g(X,\operatorname{grad}\sigma) \\ \nabla^g_X \operatorname{grad}\sigma + \rho X + \sigma \mathrm{P}^\sharp(X) \\ X(\rho) - \mathrm{P}(X,\operatorname{grad}\sigma)\end{pmatrix}.$$

Equation (1.119) provides vanishing of the last row. Also vanishing of the first row is quite obvious. Now contracting the second row with an arbitrary vector field Y gives

$$\begin{aligned}g(Y, \nabla^g_X\operatorname{grad}\sigma + \rho X + \sigma\,(P^g(X))^\sharp) &= \operatorname{Hess}^g\sigma(X,Y) + \rho g(X,Y) + \sigma P^g(X,Y) \\ &= A[g,\sigma](X,Y) = 0.\end{aligned}$$

Hence also the second row vanishes and $(\sigma, \operatorname{grad}\sigma, \rho)$ is a parallel section with respect to $\nabla^{\mathcal{T}}$. In particular it is zero everywhere or nowhere on each connected component of M. (M, g, σ) is an almost Einstein structure and so σ must not vanish at least at one point $p \in M$. Therefore $(\sigma, \operatorname{grad}\sigma, \rho)_p \neq 0$ and hence $(\sigma, \operatorname{grad}\sigma, \rho)_x \neq 0$ for all $x \in M$ such that the coincidence of $\operatorname{grad}\sigma_x = 0$ and $\sigma(x) = 0$ implies $\rho(x) \neq 0$. ■

The following proposition summarises some facts on almost Einstein structures found in [Gov10, cp. 4]. It is followed by a proof that does not use tractor calculus. Different signs in the equations compared to [Gov10] are a remainder of different conventions in the sign for the Cotton tensor.

Proposition 1.4.13. *Let (M, g, σ) be a almost Einstein structure of dimension n, then it holds*

$$\operatorname{grad}\sigma \lrcorner \mathrm{W} = \sigma\, \mathrm{C} \tag{1.125}$$
$$\sigma\mathfrak{B}(X,Y) = -(n-4)\,\mathrm{C}(X,Y,\operatorname{grad}\sigma) \tag{1.126}$$
$$(\operatorname{div}_3 \mathrm{W})(\cdot, \operatorname{grad}\sigma, \cdot) = (\operatorname{div}_2 \mathrm{W})(\cdot, \operatorname{grad}\sigma, \cdot) \tag{1.127}$$
$$\operatorname{div}\mathfrak{B} = (n-4)\operatorname{tr}^g_{1,2}\left(\operatorname{tr}^g_{1,3} \mathrm{P}\otimes \mathrm{C}\right). \tag{1.128}$$

The last equation is not special for almost Einstein structures, but holds on any manifold and is mentioned here due to its simple proof on such structures.

Proof: Throughout the proof calculations are done at $p \in M$ and using p-synchronous vector fields. By using Equation (1.111) for the divergence of the Weyl tensor one gets

$$\begin{array}{rrcl} & \operatorname{div}(\sigma^{3-n}\,\mathrm{W}) & = & 0 \\ \iff & \sigma^{2-n}\left(-(3-n)\operatorname{grad}\sigma\lrcorner \mathrm{W} + \sigma\operatorname{div}\mathrm{W}\right) & = & 0 \\ \overset{\sigma\neq 0}{\iff} & (n-3)\operatorname{grad}\sigma\lrcorner\mathrm{W} + \sigma\operatorname{div}\mathrm{W} & = & 0 \\ \overset{(1.39)}{\iff} & \operatorname{grad}\sigma\lrcorner\mathrm{W} - \sigma\,\mathrm{C} & = & 0. \end{array}$$

Since the equation holds on the dense set $M \setminus \Sigma$ it holds on the hole manifold due to the continuity of its components, which proves the first Equation (1.125).

For the second claim one can use $\sigma\mathrm{P} = -\operatorname{Hess}^g\sigma - \rho g$ to obtain

$$\begin{array}{rcl} \sigma\mathfrak{B}(X,Y) & \overset{(1.44)}{=} & -\sum\limits_{i,j}\epsilon_i\epsilon_j(\nabla\nabla\sigma)(e_i,e_j)\,\mathrm{W}(e_i,X,e_j,Y) - \sigma(\operatorname{div}_2\mathrm{C})(X,Y) \\ & \overset{(1.5)}{=} & -\sum\limits_i \epsilon_i\,\mathrm{W}(\nabla_{e_i}\operatorname{grad}\sigma, X, e_i, Y) - \sigma(\operatorname{div}_2\mathrm{C})(X,Y) \\ & \overset{(1.125)}{=} & -\sum\limits_i \epsilon_i\nabla_{e_i}\left(\sigma\,\mathrm{C}(X,e_i,Y)\right) + \sum\limits_i\epsilon_i(\nabla\mathrm{W})(e_i,\operatorname{grad}\sigma,X,e_i,Y) \\ & & -\sigma(\operatorname{div}_2\mathrm{C})(X,Y) \\ & = & -\mathrm{C}(X,\operatorname{grad}\sigma,Y) + \sigma(\operatorname{div}_2\mathrm{C}(X,Y)) \\ & & +(n-3)\,\mathrm{C}(Y,\operatorname{grad}\sigma,X) - \sigma(\operatorname{div}_2\mathrm{C})(X,Y)). \end{array}$$

Now as $\sigma\,\mathrm{C}(X,Y,\operatorname{grad}\sigma) = \mathrm{W}(\operatorname{grad}\sigma,X,Y,\operatorname{grad}\sigma)$ is symmetric in X and Y by Corollary 1.1.8, Equation (1.126) follows. This argument also proves the third claim, since by this symmetry

$$\begin{array}{rcl} (\operatorname{div}_3\mathrm{W})(X,\operatorname{grad}\sigma,Y) & = & (n-3)\,\mathrm{C}(Y,X,\operatorname{grad}\sigma) \\ & = & (n-3)\,\mathrm{C}(X,Y,\operatorname{grad}\sigma) \\ & = & (\operatorname{div}_2\mathrm{W})(Y,\operatorname{grad}\sigma,X). \end{array}$$

The divergence of the Bach tensor on almost Einstein manifolds is then derived away from the singularity set of σ. Using $\mathfrak{B}(X,\operatorname{grad}\sigma) \overset{(1.126)}{=} -\sigma^{-1}(n-4)\,\mathrm{C}(X,\operatorname{grad}\sigma,\operatorname{grad}\sigma) = 0$ and $\nabla\operatorname{grad}\sigma = -\sigma\mathrm{P}^\sharp - \rho\operatorname{id}$ gives

$$\sigma(\operatorname{div}\mathfrak{B})(X) \quad = \quad (\operatorname{div}(\sigma\mathfrak{B}))(X) + \mathfrak{B}(X,\operatorname{grad}\sigma)$$

$$
\begin{aligned}
&= -\sum_i \epsilon_i \nabla_{e_i} \left(\sigma \mathfrak{B}(e_i, X)\right) + \mathfrak{B}(X, \operatorname{grad}\sigma) \\
&\overset{(1.126)}{=} (n-4) \sum_i \epsilon_i \nabla_{e_i} \left(\mathrm{C}(e_i, X, \operatorname{grad}\sigma)\right) \\
&\overset{(1.48)}{=} (n-4) \sum_i \epsilon_i \,\mathrm{C}(e_i, X, \nabla_{e_i} \operatorname{grad}\sigma) \\
&= -(n-4)\sigma \sum_i \epsilon_i \,\mathrm{C}(e_i, X, \mathrm{P}^\sharp(e_i)) \\
&= (n-4)\sigma \sum_i \epsilon_i \,\mathrm{C}(e_i, \mathrm{P}^\sharp(e_i), X) \\
&= (n-4)\sigma \left(\operatorname{tr}^g_{1,2} \left(\operatorname{tr}^g_{1,3} \mathrm{P} \otimes \mathrm{C}\right)\right)(X).
\end{aligned}
$$

In particular $\operatorname{div}(\mathfrak{B})(X) = \operatorname{tr}^g_{1,2} \operatorname{tr}^g_{1,3} \mathrm{C}$ holds on a dense set and by continuity holds all over M. ■

Corollary 1.4.14. *Let (M, g, σ) be an almost Einstein structure, then it holds*

$$\mathrm{C}(\operatorname{grad}\sigma, \cdot, \cdot) \equiv 0 \tag{1.129}$$
$$\mathfrak{B}(\cdot, \operatorname{grad}\sigma) \equiv 0, . \tag{1.130}$$

Additional on the singularity set $\Sigma = \sigma^{-1}(0)$ one also has

$$\mathrm{C}(\cdot, \operatorname{grad}\sigma, \cdot) \equiv|_\Sigma 0 \tag{1.131}$$
$$\operatorname{grad}\sigma \lrcorner \mathrm{W} \equiv|_\Sigma 0.. \tag{1.132}$$

Proof: The first two equations are direct consequences of Equations (1.125) and (1.126) since

$$
\begin{aligned}
\mathrm{C}(\operatorname{grad}\sigma, \cdot, \cdot) &\overset{\sigma \neq 0}{=} \sigma^{-1} \mathrm{W}(\operatorname{grad}\sigma, \operatorname{grad}\sigma, \cdot, \cdot) \\
\mathfrak{B}(\cdot, \operatorname{grad}\sigma) &\overset{\sigma \neq 0}{=} (n-4)\sigma^{-1} \mathrm{C}(\cdot, \operatorname{grad}\sigma, \operatorname{grad}\sigma),
\end{aligned}
$$

where the right-hand sides vanishes due to symmetry arguments applied to Weyl and Cotton tensor. This gives correctness of (1.129) and (1.130) away from Σ. It also holds on Σ due to continuity. Finally (1.131) is a consequence of (1.126), while (1.132) is implied by (1.125) if σ is sent to zero. ■

So one in particular has $(n-4)\,\mathrm{W}(\operatorname{grad}\sigma, X, Y, \operatorname{grad}\sigma) = -\sigma^2 \mathfrak{B}(X, Y)$ on almost Einstein structures. By considering Equation (1.45) for the Bach tensor, the following consequence can be obtained.

Corollary *Let (M, g, σ) be an almost Einstein structure, then*

$$
\begin{aligned}
&(n-4)\,\mathrm{W}(\operatorname{grad}\sigma, X, Y, \operatorname{grad}\sigma) = \\
&\quad \sigma^2 \left((\Delta \mathrm{P} + \operatorname{Hess}^g J)(X, Y) + n\mathrm{P}(X, \mathrm{P}^\sharp(Y)) - \|\mathrm{P}\|^2_g g(X, Y) - 2 \operatorname{tr}^g_{1,3}(\operatorname{tr}^g_{1,3} \mathrm{P} \otimes \mathrm{W}) \right).
\end{aligned} \tag{1.133}
$$

1.5 MORSE LEMMA

Let $f : M \to \mathbb{R}$ be a C^2 map on M. This section will recall two important facts that help to either get more information on the local topology of M or to locally characterise the behaviour of f. The notation will be fixed first. A point $p \in M$ is a *non-degenerate critical point* of f if $df_p = 0$ and at the same time the Hessian is non-singular. The maximal dimension of a subspace for which $\operatorname{Hess} f_p$ is negative definite is the *index* of $\operatorname{Hess} f_p$. A map $f : X \to Y$ on topological spaces X and Y is called proper if $f^{-1}(C) \subset X$ is compact for all compact subsets $C \subset Y$. A subset Y of a topological space X is called *deformation retract* of X, if there is a continuous map $F : X \times [0,1] \to X$ such that

$$F(\cdot, 0) = \mathrm{id} \qquad F(X,1) \subset Y \qquad F(y,1) =|_Y \mathrm{id}\,.$$

Theorem 1.5.1. *[Mil63, Theorem 3.1] Let $f : M^n \to \mathbb{R}$ be a smooth function on an n-dimensional manifold M^n. Suppose that the set $f^{-1}([a,b])$ is compact and contains no critical points of f for $a < b \in \mathbb{R}$. Then $f^{-1}((-\infty, a))$ is diffeomorphic to $f^{-1}((-\infty, b))$ and $f^{-1}((-\infty, a))$ is deformation retract of $f^{-1}((-\infty, b))$.*

Lemma 1.5.2. **(***Morse Lemma***)** *[Mil63, Lemma 2.2] Consider a smooth n-dimensional manifold M. Let $p \in M$ be a non-degenerate critical point of $f : M^n \to \mathbb{R}$ and α be the index of* Hess f_p*. Then there is a neighbourhood U of p and a chart (U, φ) with $\varphi(p) = 0$ such that*

$$f(x) = f(p) - \sum_{i=1}^{\alpha} \left(\varphi^i(x)\right)^2 + \sum_{i=\alpha+1}^{n} \left(\varphi^i(x)\right)^2 \tag{1.134}$$

for all $x \in U$.

Let p be such a non-degenerate critical point. Then the Morse lemma provides a neighbourhood, such that f may be written in the above form and its differential reads as

$$df_x = -2\sum_{i=1}^{\alpha} \varphi^i(x) d\varphi^i_x + \sum_{i=\alpha+1}^{n} \varphi_i(x) d\varphi^i_x.$$

The coordinate differentials $d\varphi^i$ are linearly independent for all $x \in U$ and one finds the equality $df_x = 0 \quad \Leftrightarrow \quad \varphi^i(x) = 0 \ \forall i \in \{1, \dots, n\} \quad \Leftrightarrow \quad x = p$. This provides an additional property of such points.

Corollary 1.5.3. *[Mil63, Corollary 2.3] Non-degenerate critical points are isolated in the set of critical points.*

2 EXAMPLES OF ALMOST EINSTEIN STRUCTURES

2.1 EMBEDDINGS IN THE PSEUDOSPHERE

In this section conformal embeddings of pseudo-Euclidean space, anti-de Sitter and de Sitter space into the pseudosphere are presented. Those three spaces can be seen as basic models for Einstein manifolds. The conformal embedding then is interpreted as an almost Einstein structure on the pseudosphere. Each space is a model for a different sign of $S[g,\sigma]$. For each embedding, its properties in a neighbourhood of the conformal boundary are discussed.

2.1.1. Construction of a Pseudosphere

First the construction of a *pseudosphere* will be given. For that consider the $(n+2)$-dimensional pseudo-Euclidean space $\mathbb{R}^{p+1,q+1} := \left(\mathbb{R}^{n+2}, \langle , \rangle_{p+1,q+1}\right)$ with $p,q > 0$, $p+q = n$ and null cone

$$\mathcal{C}^{p+1,q+1} = \{x \in \mathbb{R}^{p+1,q+1} \mid \|x\|^2_{p+1,q+1} = 0\}.$$

Here the short notation $\|x\|^2_{p+1,q+1} := \langle x, x\rangle_{p+1,q+1} = -\left(x^0\right)^2 - \cdots - (x^p)^2 + \left(x^{p+1}\right)^2 + \cdots + \left(x^{n+1}\right)^2$ is used. For the rest of this chapter the lower index on the quadratic form $\| \|^2_{i,j}$ will indicate the scalar product that is used to build it. Intersection of $\mathcal{C}^{p+1,q+1}$ with the $(n+1)$-dimensional sphere of radius $\sqrt{2}$, $S^{n+1}_{\sqrt{2}} = \left\{x \in \mathbb{R}^{p+1,q+1} \mid \|x\|^2_{n+2} = 2\right\}$ is denoted by

$$S^{p,q} := \mathcal{C}^{p+1,q+1} \cap S^{n+1}_{\sqrt{2}}.$$

Since $p,q > 0$ is required, this submanifold of $\mathbb{R}^{p+1,q+1}$ is connected. Let g be the metric that is induced by $\langle , \rangle_{p+1,q+1}$. Then the pseudo-Riemannian submanifold $(S^{p,q}, g)$ is called the *pseudosphere of index* (p,q).

By considering sum and difference of $\|x\|^2_{n+2} = 2$ and $\|x\|^2_{p+1,q+1} = 0$ the submanifold can be written as

$$S^{p,q} = \left\{x \in \mathbb{R}^{p+1,q+1} \mid \left(\left(x^0\right)^2 + \cdots + \left(x^p\right)^2 = 1\right) \wedge \left(\left(x^{p+1}\right)^2 + \cdots + \left(x^{n+1}\right)^2 = 1\right)\right\}$$

and hence is diffeomorphic to $S^p \times S^q$. The canonic projection maps are denoted by

$$\begin{aligned}\pi_1 &: S^{p,q} \to S^p \subset \mathbb{R}^{p+1}\\ \pi_2 &: S^{p,q} \to S^q \subset \mathbb{R}^{q+1}\end{aligned}$$

such that the induced metric g may be written as pullback of round metrics

$$g = -\pi_1^* g_{S^p} + \pi_2^* g_{S^q}. \tag{2.1}$$

It will be denoted $g = -g_{S^p} + g_{S^q}$ later without taking care of the pullback each time.

2.1.2. Pseudo-Euclidean Space

The first example of a conformally embedded manifold in the pseudosphere is the pseudo-Euclidean space $\mathbb{R}^{p,q}$. The map

$$\begin{array}{rccc}\iota: & \mathbb{R}^{p,q} & \longrightarrow & S^{p,q}\\ & \hat{x} & \longmapsto & \pi_{S^{n+1}_{\sqrt{2}}}\left(1 + \langle\hat{x},\hat{x}\rangle_{p,q}, 2\hat{x}, 1 - \langle\hat{x},\hat{x}\rangle_{p,q}\right).\end{array} \tag{2.2}$$

maps the pseudo-Euclidean space $\mathbb{R}^{p,q}$ to a subset of the pseudosphere. The projection to the sphere $S^{n+1}_{\sqrt{2}}$ of radius $\sqrt{2}$ is done by rescaling. To show that this indeed is an embedding is the intention of the following paragraphs. First basic properties of $\imath$ and its inverse will be given.

The point $x = \left(1 + \langle \hat{x}, \hat{x} \rangle_{p,q}, 2\hat{x}, 1 - \langle \hat{x}, \hat{x} \rangle_{p,q}\right)$ has squared Euclidean norm $\|x\|^2_{n+2} = 2\left(1 + 2\|\hat{x}\|^2_n + \langle \hat{x}, \hat{x} \rangle^2_{p,q}\right)$ and hence projection to the sphere of radius $\sqrt{2}$ fixes the scale factor to

$$N(\hat{x}) := \left(1 + 2\|\hat{x}\|^2_n + \langle \hat{x}, \hat{x} \rangle^2_{p,q}\right)^{1/2}, \tag{2.3}$$

or in other words $\imath(\hat{x}) = \frac{1}{N(\hat{x})}\left(1 + \langle \hat{x}, \hat{x} \rangle_{p,q}, 2\hat{x}, 1 - \langle \hat{x}, \hat{x} \rangle_{p,q}\right)$. In particular $N(\hat{x}) \geq 1$ and hence it is a positive number. Consequently $\imath$ is the corestriction to the codimension 2 submanifold $S^{p,q}$ of a C^1 map with values in $\mathbb{R}^{p+1,q+1}$ and hence it is a C^1-map. With the definition $\sigma : \mathbb{R}^{p+1,q+1} \ni x \mapsto x^0 + x^{n+1} \in \mathbb{R}$, now the inverse of $\imath$ is provided by the restriction of

$$\begin{array}{rccc} \imath^{-1}: & \mathbb{R}^{p+1,q+1} \setminus \sigma^{-1}(0) & \longrightarrow & \mathbb{R}^{p,q} \\ & x = (x^0, \hat{x}, x^{n+1}) & \longmapsto & \frac{\hat{x}}{\sigma(x)} \end{array}$$

to the image of $\imath$. The zero set $\sigma^{-1}(0)$ is a hyperplane in $\mathbb{R}^{n+2}$, which makes the domain of $\imath^{-1}$ a submanifold and $\imath^{-1}$ a smooth map on it. The latter remark remains true, if $\imath^{-1}$ is further restricted to $S^{p,q} \setminus \sigma^{-1}(0)$. A direct calculation (Lemma A.1.6) shows $\imath^{-1} \circ \imath = \mathrm{id}_{\mathbb{R}^{p,q}}$. This in particular ensures injectivity of $d\imath_{\hat{x}}$ for all $\hat{x} \in \mathbb{R}^{p,q}$. The differential of the inverse mapping at $x \in \imath\left(\mathbb{R}^{p,q}\right) \subset S^{p,q}$ will be needed later and can easily be calculated to

$$\begin{array}{rccc} d\imath^{-1}_x: & T_x S^{p,q} & \longrightarrow & T\mathbb{R}^{p,q} \simeq \mathbb{R}^n \\ & (v^0, \hat{v}, v^{n+1}) & \longmapsto & \frac{1}{\sigma(x)}\left(\hat{v} - \sigma(v)\imath^{-1}(x)\right). \end{array} \tag{2.4}$$

As $\sigma \circ \imath\left(\hat{x}\right) = \frac{2}{N(\hat{x})} \neq 0$, this is well defined and continuously depends on x for all $x \in \imath\left(\mathbb{R}^{p,q}\right)$. One finally gets the following corollary.

Corollary 2.1.1. *The map $\imath$, as given in Equation (2.2) is an embedding of the pseudo-Euclidean space into the pseudosphere.*

Proof: Due to the preceding calculations the inclusion map $\imath$ is a injective smooth map with injective differential, which makes $\mathrm{im}(\imath)$ an immersed submanifold of $S^{p,q}$. On the other hand continuity of $\imath^{-1}|_{S^{p,q}\setminus\sigma^{-1}(0)}$ with respect to the subspace topology inherited from $S^{p,q}$ ensures that $\imath$ also is a topological embedding, which finally proves the claim. ∎

Important to the interpretation of the pseudosphere as almost Einstein structure is the pullback of the pseudo-Euclidean flat metric under the given embedding. The flat metric $\langle , \rangle_{p,q}$ of signature (p, q) will also be denoted $g_{p,q}$. The notation is useful to evidently distinguish between the metric and the scalar product that is used to do the calculations.

Lemma 2.1.2. *The pullback of the flat metric $g_{p,q}$ to the pseudosphere $(S^{p,q}, g) \subset \left(\mathbb{R}^{p+1,q+1}, g_{p+1,q+1}\right)$ under $\imath^{-1}$ is given by*

$$\imath^{-1*} g_{p,q} = \sigma^{-2}\, g_{p+1,q+1}\big|_{TS^{p,q}} \tag{2.5}$$

where $g_{p+1,q+1}\big|_{TS^{p,q}} = -g_{S^p} + g_{S^q}$.

Proof: Helpful to the proof of this lemma is that at least at a neighbourhood of each $y \in \imath\left(\mathbb{R}^{p,q}\right) \subset \mathbb{R}^{p+1,q+1}$ the map $\imath^{-1}$ has a well defined generic extension. A direct corollary is that the pullback fulfils $\imath^{-1*} g_{p,q} = \sigma^{-2} g$, where g is the submanifold metric on the pseudosphere.

First of all one observes the following fact. Consider $x \in S^{p,q} \subset \mathbb{R}^{n+2}$ and a vector $v \in T_x S^{p,q} \subset \mathbb{R}^{n+2}$ with the canonical identification. Then $v \in T_x(S^p \times S^q)$, as the pseudosphere is a submanifold of $\mathbb{R}^{p+1} \times \mathbb{R}^{q+1}$. Hence $x^0 v^0 + \cdots + x^p v^p = 0$ and $x^{p+1} v^{p+1} + \cdots + x^{n+1} v^{n+1} = 0$. Now writing $x = (x^0, \hat{x}, x^{n+1})$ and in a similar way $v = (v^0, \hat{v}, v^{n+1})$ gives

$$\begin{aligned} \langle \hat{x}, \hat{v} \rangle_{p,q} &= \langle x, v \rangle_{p+1,q+1} + x^0 v^0 - x^{n+1} v^{n+1} \\ &= x^0 v^0 - x^{n+1} v^{n+1}. \end{aligned} \tag{2.6}$$

On the other hand $x \in \mathcal{C}^{p+1,q+1}$ is an element of the null cone and consequently

$$\begin{aligned} \langle \hat{x}, \hat{x} \rangle_{p,q} &= \langle x, x \rangle_{p+1,q+1} + \left(x^0 + x^{n+1}\right)\left(x^0 - x^{n+1}\right) \\ &= \sigma(x)(x^0 - x^{n+1}). \end{aligned} \tag{2.7}$$

Also by Equation (2.4) one has

$$\sigma(x) d\imath_x^{-1}(v) = \hat{v} - \frac{\sigma(v)}{\sigma(x)} \hat{x}. \tag{2.8}$$

Now the pullback metric can be calculated. Let $x \in S^{p,q}$ be a point and $v, w \in T_x S^{p,q}$ tangent vectors, which will be canonically interpreted as elements of $\mathbb{R}^{n+2}$. Then

$$\begin{aligned} \sigma^2(x)\left(\imath^{-1*} g_{p,q}\right)_x (v,w) &= \left\langle \sigma(x) d\imath_x^{-1}(v), \sigma(x) d\imath_x^{-1}(w) \right\rangle_{p,q} \\ &\overset{(2.8)}{=} \langle \hat{v}, \hat{w} \rangle_{p,q} + \frac{\sigma(v)\sigma(w)}{\sigma^2(x)} \langle \hat{x}, \hat{x} \rangle_{p,q} \\ &\quad - \frac{\sigma(v)}{\sigma(x)} \langle \hat{x}, \hat{w} \rangle_{p,q} - \frac{\sigma(w)}{\sigma(x)} \langle \hat{x}, \hat{v} \rangle_{p,q} \\ &\overset{(2.6),(2.7)}{=} \langle \hat{v}, \hat{w} \rangle_{p,q} + \frac{\sigma(v)\sigma(w)}{\sigma(x)} \left(x^0 - x^{n+1}\right) \\ &\quad - \frac{\sigma(v)}{\sigma(x)} \left(x^0 w^0 - x^{n+1} w^{n+1}\right) \\ &\quad - \frac{\sigma(w)}{\sigma(x)} \left(x^0 v^0 - x^{n+1} v^{n+1}\right) \\ &= \langle \hat{v}, \hat{w} \rangle_{p,q} + \frac{\left(-v^0 w^0 + v^{n+1} w^{n+1}\right)\left(x^0 + x^{n+1}\right)}{\sigma(x)} \\ &= \langle v, w \rangle_{p+1,q+1}. \end{aligned}$$

■

This immediately leads to the following generalisation.

Corollary 2.1.3. *$(S^{p,q}, g, \sigma)$ is an almost Einstein structure with $S[g,\sigma] = 0$.*

Proof: Consider the map

$$\begin{array}{cccc} f: & \mathbb{R}^{n+2} & \longrightarrow & \mathbb{R}^3 \\ & (x^0, \hat{x}, x^{n+1}) & \longmapsto & \begin{pmatrix} (x^0)^2 + \cdots + (x^p)^2 - 1 \\ (x^{p+1})^2 + \cdots + (x^{n+1})^2 - 1 \\ x^0 + x^{n+1} \end{pmatrix}^t \end{array}$$

The singularity set is $\Sigma = \left\{ x \in \mathbb{R}^{n+2} \mid f(x) = 0 \right\}$. df_x is not of full rank at points $x \in f^{-1}(0)$ where $\|x^0\| = \|x^{n+1}\| = 1$, since at such points the covectors $x^0 dx^0 + \cdots + x^p dx^p$, $x^{p+1} dx^{p+1} + \cdots + x^{n+1} dx^{n+1}$ and $dx^0 + dx^{n+1}$ are collinear. There are two such points, which will be given later on. After removing these points from Σ, the remaining set is a $(n-1)$ dimensional submanifold of $\mathbb{R}^{n+2}$ by the regular value theorem. Consequently Σ is a $(n-1)$ dimensional submanifold of $S^{p,q}$, except at those two points and hence $S^{p,q} \setminus \Sigma$ is a dense subset. It suffices to show that $\sigma^{-2} g$ is an Einstein metric with scalar curvature $\tau = 0$ at that dense subset. Due to the previous lemma this is clear for points $x \in S^{p,q}$, where $\sigma(x) > 0$. In case of negative σ one may consider a modified embedding of the pseudo-Euclidean space into the pseudosphere, given by $\bar{\imath}(\hat{x}) = -\imath(\hat{x})$. The formula for the inverse function is the same as before, but it will be denoted $\bar{\imath}^{-1}$ due to the different domain it is defined on. Then by repeating the calculations that lead to the previous lemma one obtains

$$\bar{\imath}^{-1*} g_{p,q} = \sigma^{-2} \, g_{p+1,q+1} \big|_{TS^{p,q}}. \tag{2.9}$$

This again is an Einstein metric with vanishing scalar curvature. It remains to show that $\imath\left(\mathbb{R}^{p,q}\right) \cup \bar{\imath}\left(\mathbb{R}^{p,q}\right) = S^{p,q} \setminus \Sigma$. To show this one may consider $\imath^{-1}$ as map with domain $S^{p,q}$ and codomain $\mathbb{R}^{p,q}$. Now by Lemma A.1.6 one has

$$\imath \circ \imath^{-1}\left(x^0, \hat{x}, x^{n+1}\right) = \pi_{S^{n+1}_{\sqrt{2}}}\left(\sigma(x)\left(x^0 + 0, \hat{x}, x^{n+1} - 0\right)\right)$$

for $x = \left(x^0, \hat{x}, x^{n+1}\right) \in S^{p,q} \setminus \sigma^{-1}(0)$ and equivalently

$$\bar{\imath} \circ \imath^{-1}\left(x^0, \hat{x}, x^{n+1}\right) = -\pi_{S^{n+1}_{\sqrt{2}}}\left(\sigma(x)\left(x^0, \hat{x}, x^{n+1}\right)\right).$$

The first composition gives the identity where $\sigma(x) > 0$, while the second composition gives the identity, where $\sigma(x) < 0$. Hence $\bar{\imath}$ and $\imath$ cover the pseudosphere except for point with vanishing σ.

This shows that the almost Einstein tensor vanishes on the dense set $S^{p,q} \setminus \Sigma$ and hence vanishes all over $S^{p,q}$, such that $\left(S^{p,q}, g, \sigma\right)$ is an almost Einstein structure. Vanishing almost scalar curvature is a consequence of vanishing scalar curvature for the pseudo-Euclidean space and Equation (1.124). ■

Conformal Boundary

Next the causal structure of the conformal boundary Σ is discussed in the case of the above almost Einstein structure on the pseudosphere. In case of signature $(p, q) = (1, n-1)$ this topic is well studied in literature (see for example [HE73, O'N83, Pen11]). It will be summarised here due to its importance to this thesis as a toy model.

Initial points of the subsequent observations are geodesics in pseudo-Euclidean space $\mathbb{R}^{p,q}$ and their limits after mapped to the pseudosphere. The 3 causal types of geodesics $\gamma : \mathbb{R} \to \mathbb{R}^{p,q}$ are denoted by

$$\begin{aligned} &\text{(i)} && \gamma_n(t) = \hat{x} + t\hat{v}_n \\ &\text{(ii)} && \gamma_s(t) = \hat{x} + t\hat{v}_s \\ &\text{(iii)} && \gamma_t(t) = \hat{x} + t\hat{v}_t. \end{aligned}$$

Here v_n is an arbitrary null vector, v_s is an arbitrary spacelike vector and v_t a timelike vector. Hence the corresponding geodesics are null, spacelike and timelike. The curves $\imath \circ \gamma$ or $\bar{\imath} \circ \gamma$ still are geodesics on the pseudosphere with respect to the metric $\sigma^{-2}g$. Before taking the limit $t \to \infty$ one observes

$$\begin{aligned} (i) \quad \left\langle \gamma_n(t), \gamma_n(t) \right\rangle_{p,q} &= \left\langle \hat{x}, \hat{x} \right\rangle_{p,q} + 2tr \\ (ii) \quad \left\langle \gamma_s(t), \gamma_s(t) \right\rangle_{p,q} &= \left\langle \hat{x}, \hat{x} \right\rangle_{p,q} + 2t\left\langle \hat{x}, \hat{v}_s \right\rangle_{p,q} + t^2\left\langle \hat{v}_s, \hat{v}_s \right\rangle_{p,q} \\ (iii) \quad \left\langle \gamma_s(t), \gamma_t(t) \right\rangle_{p,q} &= \left\langle \hat{x}, \hat{x} \right\rangle_{p,q} + 2t\left\langle \hat{x}, \hat{v}_t \right\rangle_{p,q} + t^2\left\langle \hat{v}_t, \hat{v}_t \right\rangle_{p,q} \end{aligned}$$

where $r = \left\langle \hat{x}, \hat{v}_n \right\rangle_{p,q}$. Now the limits are

$$\begin{aligned} \lim_{t\to\infty} \imath \circ \gamma_n(t) &= \pi_{S^{n+1}_{\sqrt{2}}}\left(r, \hat{v}_n, -r\right) \\ \lim_{t\to\infty} \imath \circ \gamma_s(t) &= (1, \hat{0}, -1) \\ \lim_{t\to\infty} \imath \circ \gamma_t(t) &= (-1, \hat{0}, 1). \end{aligned}$$

In case where the embedding $\bar{\imath}$ is used the latter two points are interchanged. It is clear that the limit points belong to $\Sigma = \sigma^{-1}(0)$. In signature $(p, q) = (1, n-1)$ the property of being the limit point for spacelike or timelike geodesics motivates the term *spacelike* or *timelike infinity* for $(1, \hat{0}, -1)$ and $(-1, \hat{0}, 1)$. As a matter of fact, those two points are the two points, where the regular value theorem used in the proof of Corollary 2.1.3 fails. Timelike and spacelike infinity interchange its roles, if the embedding $\bar{\imath}$ is used. The remaining part

$\Sigma \setminus \{(1, \hat{0}, -1), (-1, \hat{0}, 1)\}$ is called *null infinity* since it is the limit point of null geodesics of $\sigma^{-2}g$. In signature $(p, q) = (1, n-1)$ it has the property to decompose into two disjoint components. One is generated by all $\hat{v}_n$ with negative first component and one generated by all $\hat{v}_n$ with positive first component. There cannot exist a path from one point in the first set to a point in the second set, since it would have a zero-crossing in its first component and hence this would imply $\hat{v}_n = 0$. This argumentation does not hold any longer in signatures, where $p, q > 1$.

In principle at this point it is not guaranteed that the limit points of null geodesics do not just generate a smaller subset of $\Sigma \setminus \{(1, \hat{0}, -1), (-1, \hat{0}, 1)\}$. But this is provided by the fact that $\langle \hat{y}, \hat{y} \rangle = 0$ for all $(y^0, \hat{y}, y^{n+1}) \in \Sigma$. Choosing $\hat{v}_n = \hat{y}$ then gives the claimed inclusion in the opposite direction. Consequently the topological boundary $\partial\imath(\mathbb{R}^{p,q}) \subset S^{p,q}$ and the locus Σ of σ coincide

$$\Sigma = \partial\imath(\mathbb{R}^{p,q}). \tag{2.10}$$

Next the gradient of σ in $(x^0, \hat{x}, x^{n+1}) = x \in S^{p,q}$ will be derived with respect to the metric g of the pseudosphere. As submanifold of $(\mathbb{R}^{n+2}, g_{p+1,q+1})$ this is provided by the projection of $\operatorname{grad}_x^{g_{p+1,q+1}} \sigma = (-1, \hat{0}, 1)$ to the tangent space $T_x S^{p,q}$ if taken with respect to $g_{p+1,q+1}$. Since this means, to project the first $p+1$ components to the tangent space of S^p in $(x^0, \ldots, x^p)$ and the last $q+1$ components to the tangent space of S^q in $(x^{p+1}, \ldots, x^{n+1})$ the result is

$$\operatorname{grad}_x^g \sigma = \left(-1 + \left(x^0\right)^2, x^0x^1, \ldots, x^0x^p, -x^{n+1}x^{p+1}, \ldots, -x^{n+1}x^n, 1 - \left(x^{n+1}\right)^2\right).$$

In particular

$$\begin{aligned} g\left(\operatorname{grad}_x^g \sigma, \operatorname{grad}_x^g \sigma\right) &= -\left(x^0\right)^2 \left(\left(x^0\right)^2 + \cdots + (x^p)^2 - 2\right) \\ &\quad + \left(x^{n+1}\right)^2 \left(\left(x^{p+1}\right)^2 + \cdots + \left(x^{n+1}\right)^2 - 2\right) \\ &= \left(x^0\right)^2 - \left(x^{n+1}\right)^2 \\ &= \sigma(x)\left(x^0 - x^{n+1}\right) \end{aligned}$$

vanishes where σ does, which one would expect anyhow by Equation (1.115), since $S[g, \sigma] = 0$ holds for the almost Einstein structure under consideration. On the other hand the explicit form is interesting, since it obviously shows that the quotient $\frac{g(\operatorname{grad}\sigma, \operatorname{grad}\sigma)}{\sigma^k}$ has only a continuous extension to Σ for $k \leq 1$.

Parametrising a Neighbourhood of Timelike Infinity

Next in case of Lorentzian signature $(1, n-1)$ for the pseudosphere special coordinates $\varphi^{-1} : U \subset \mathbb{R}^n \to S^{1,n-1}$ are constructed in a neighbourhood of conformal timelike infinity $(-1, \hat{0}, 1)$. The aim is to construct them such that they are sufficiently smooth and such that in these coordinates up to a constant factor c one has $\sigma \circ \varphi^{-1}(y) = c\langle y, y \rangle_{1,n-1}$. By construction then also null lines on the Minkowski null cone in $\mathbb{R}^{1,n-1} \simeq \mathbb{R}^n$ are mapped to null curves in $S^{1,n-1}$.

Let

$$\begin{aligned} \psi^{-1} : \quad U \subset \mathbb{R}^n &\longrightarrow S^{1,n-1} \\ \hat{x} &\longmapsto \pi_{S^{n+1}_{\sqrt{2}}}\left(-16 - \langle \hat{x}, \hat{x} \rangle_{1,n-1}, 8\hat{x}, 16 - \langle \hat{x}, \hat{x} \rangle_{1,n-1}\right) \end{aligned}$$

be coordinates for a small neighbourhood of $(-1, \hat{0}, 1)$. Using $N(\hat{x}) = \frac{1}{\sqrt{2}} \left\| \psi^{-1}(\hat{x}) \right\|_{n+2} = \left(256 + \langle \hat{x}, \hat{x} \rangle_{1,n-1}^2 + \|\hat{x}\|^2\right)^{1/2}$ one may write ψ as

$$\psi^{-1}(\hat{x}) = \frac{1}{N(\hat{x})}\left(-16 - \langle \hat{x}, \hat{x} \rangle_{1,n-1}, 8\hat{x}, 16 - \langle \hat{x}, \hat{x} \rangle_{1,n-1}\right).$$

Those coordinates are C^∞-smooth in a neighbourhood of $\hat{0}$. Also null lines parametrised by $t \mapsto t(1, \mathbf{e})$, where $\mathbf{e}$ is a unit vector in $\mathbb{R}^{n-1}$ are map to curves whose tangent vector is a linear combination of $(-16, 1, \mathbf{e}, 16)$ and $(0, 1, \mathbf{e}, 0)$. The latter vectors span a 2-dimensional totally isotropic subspace at the tangent space and hence the corresponding curve is a null curve. In particular an additional reparametrisation will not change that property.

For σ one finds

$$\sigma\left(\psi^{-1}(\hat{x})\right) = -2\frac{\langle \hat{x}, \hat{x}\rangle_{1,n-1}}{N(\hat{x})},$$

which is the desired result up to a scalar function. On the other hand if $\hat{x}$ is multiplied by a constant factor $\kappa \in \mathbb{R}$, then

$$\sigma\left(\psi^{-1}(\kappa\hat{x})\right) = -2\frac{\kappa^2}{N(\kappa\hat{x})}\langle \hat{x}, \hat{x}\rangle_{1,n-1}.$$

Defining $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ by

$$\begin{aligned} f(\kappa, \hat{x}) &:= \kappa^4 - N^2(\kappa\hat{x}) \\ &= \kappa^4\left(1 - \langle \hat{x}, \hat{x}\rangle_{1,n-1}^2\right) - 32\kappa^2\langle \hat{x}, \hat{x}\rangle_n - 256 \end{aligned}$$

gives a polynomial in κ with smooth coefficients in $\hat{x}$, $f(\hat{0}, 4) = 0$ and with partial derivative $\partial_\kappa f\left(\hat{0}, 4\right) = 4^4 \neq 0$. Hence by implicit function theorem[1], there is a neighbourhood of $\hat{0}$ and a smooth map $\kappa : U \subset \mathbb{R}^n \to \mathbb{R}$ with $\kappa(\hat{0}) = 4$, such that $f(\hat{x}, \kappa(\hat{x})) = 0$. In particular this locally in a neighbourhood of the origin implies

$$\frac{\kappa^2(\hat{x})}{N(\kappa(\hat{x})\,\hat{x})} \equiv 0.$$

The desired coordinates are then given by

$$\begin{array}{rccc} \varphi^{-1}: & U \subset \mathbb{R}^n & \longrightarrow & S^{1,n-1} \\ & \hat{x} & \mapsto & \psi^{-1}(\kappa(\hat{x})\,\hat{x}) \end{array}$$

in some neighbourhood U of the origin. This map is C^∞-smooth, since it is a composition of such maps. And it indeed is locally bijective, since its differential at $\hat{0}$ is $d\varphi_{\hat{0}}^{-1} = 4d\psi_{\hat{0}}^{-1}$ and has full rank. By the previous considerations it holds that $\sigma\left(\varphi^{-1}(\hat{x})\right) = -2\langle \hat{x}, \hat{x}\rangle_{1,n-1}$, which is the desired result. As it was mentioned in the construction one also has null lines at the Minkowskian null cone to be mapped to null curves in $S^{1,n-1}$. In contrast to the coordinates that will be constructed in a more general setting later on in this thesis, the map φ does not lack to be smooth at the origin. So the open question remains whether the smoothness could also be preserved in the general setting by adjusting the construction.

2.1.3. De Sitter Space

The de Sitter space is another important model space. It is an Einstein manifold with constant positive scalar curvature. Aim of this section is to give an embedding of the de Sitter space of signature (p, q) into the pseudosphere. The calculations are basic and just by modifying them slightly this will give an embedding of a closely related different model space into the pseudosphere, the anti-de Sitter space.

Consider the pseudo-Euclidean space $\left(\mathbb{R}^{n+1}, g^{p,q+1}\right)$ with $n = p + q$.

Definition 2.1.4. The *de Sitter space* of signature (p, q) is the embedded submanifold of $\mathbb{R}^{p,q+1}$

$$dS^{p,q} = \{x \in \mathbb{R}^{n+1} \mid \langle x, x\rangle_{p,q+1} = 1\}$$

together with the induced metric, denoted by g^{dS}.

1 This particular problem is the second example of section 3.4 in [Kön04].

The signature is suppressed in the notation of the metric but will be (p,q) throughout the section. Before specifying an embedding into the pseudosphere, its model character is characterised by the following lemma.

Lemma 2.1.5. *The submanifold* $\left(dS^{p,q}, g^{dS}\right)$ *is Einstein with scalar curvature* $\tau^{dS} = n(n-1)$.

Proof: A basic proof is given in [O'N83, Chapter 4]. The tangent space $T_x dS^{p,q}$ is canonically identified with a subspace of $\mathbb{R}^{n+1}$. Then the position vector field $P_x := \sum_i x^i \partial_i$ restricted to $dS^{p,q}$ is the unit normal of the de Sitter space, in particular $g^{p,q+1}(P,P) = \langle x, x\rangle_{p,q+1} =|_{dS^{p,q}} 1$ and $g^{p,q+1}(P,V) \equiv|_{dS^{p,q}} = 0$ for vector fields that are tangent to the de Sitter space. The second statement is a consequence of the fact that for curves γ on $dS^{p,q}$ it holds $\langle \dot\gamma, \gamma\rangle_{p,q+1} \equiv 0$. Also for the position vector field one finds $\nabla^{p,q+1}_X P = X$, where $\nabla^{p,q+1}$ is the flat Levi-Civita connection on $\mathbb{R}^{p,q+1}$ and X is a vector field thereon. Now the shape operator of $dS^{p,q}$ is $S(X) = -\nabla^{dS}_X P = -X$ and hence negative of the identity. Consequently the de Sitter space has constant sectional curvature 1 and therefore $\mathrm{Ric}^{dS}(X,X) = (n-1)\, g^{p,q+1}|_{TdS}(X,X)$. Polarisation then gives $\mathrm{Ric}^{dS} = (n-1)g^{dS}$. ∎

The de Sitter space can be embedded into the pseudosphere $S^{p,q}$ by the map

$$\begin{array}{cccc} \imath: & dS^{p,q} & \longrightarrow & S^{p,q} \\ & x = (x^1, \ldots, x^{n+1}) & \longmapsto & \pi_{S^{n+1}_{\sqrt{2}}}\left(1, x^1, \ldots, x^{n+1}\right). \end{array} \tag{2.11}$$

In particular the first component will be non-zero, which is why the inverse mapping can easily be given by restricting the map

$$\begin{array}{cccc} \imath^{-1}: & \mathbb{R}^{p+1,q+1} \setminus \sigma^{-1}(0) & \longrightarrow & dS^{p,q} \\ & (x^0, \ldots, x^{n+1}) & \longmapsto & \frac{1}{\sigma(x)}(x^1, \ldots, x^{n+1}), \end{array} \tag{2.12}$$

to the image $\imath\left(dS^{p,q}\right)$, where this time the boundary defining function is

$$\sigma(x) = x^0.$$

For the next calculation $\sigma : \mathbb{R}^{n+2} \to \mathbb{R}$ is also used for vectors by the generic identification $T_x\mathbb{R}^{n+2} \simeq \mathbb{R}^{n+2}$.

Lemma 2.1.6. $\imath$ *indeed is an embedding.*

Proof: The proof essentially coincides with that for the embedding of the pseudo-Euclidean space into the pseudo-sphere. At first one finds that the restriction of $\imath^{-1}$ to the image of $\imath$ indeed gives the inverse map. Next one observes that $\imath$ is a homeomorphism, as its inverse is the restriction of a continuous map on $\mathbb{R}^{p+1,q+1} \setminus \sigma^{-1}(0)$. Finally it is an immersion, as $\imath$ and $\imath^{-1}$ are smooth maps and hence $d\imath$ must be injective everywhere. ∎

Lemma 2.1.7. *The pullback of the induced metric* $g^{dS} := g_{p+1,q}|_{dS}$ *to the pseudosphere* $(S^{p,q}, g) \subset (\mathbb{R}^{p+1,q+1}, g_{p+1,q+1})$ *is given by*

$$\imath^{-1*} g^{dS} = \sigma^{-2}\, g_{p+1,q+1}\,\big|_{TS^{p,q}}. \tag{2.13}$$

Proof: The proof is analogous to the one carried out in the case of the pseudo-Euclidean space. Let $x \in S^{p,q}$ be a point and $v, w \in T_x S^{p,q}$ tangent vectors, which will be canonically interpreted as elements of $\mathbb{R}^{n+2}$. Writing $x = (x^0, \hat x)$, $v = (v^0, \hat v)$ and $w = (w^0, \hat w)$ one may write the exterior derivative of $\imath^{-1}$ at x as

$$d\imath^{-1}_x(v) = \frac{1}{\sigma(x)}\left(\hat v - v^0 \imath^{-1}(x)\right). \tag{2.14}$$

Moreover it holds

$$\langle \hat{x}, \hat{v} \rangle_{p,q+1} = \sigma(x) v^0 \tag{2.15}$$
$$\langle \hat{x}, \hat{x} \rangle_{p,q+1} = \sigma^2(x) \tag{2.16}$$

where the first formula holds due to $v \in T_x S^{p,q}$. The calculations, which lead to the above equations are analogue to the ones carried out in the case of the pseudo-Euclidean space. Then

$$\begin{aligned}
\sigma^2(x) \left(\iota^{-1*} g^{dS}\right)_x (v, w) &= \left\langle \sigma(x) d\iota_x^{-1}(v), \sigma(x) d\iota_x^{-1}(w) \right\rangle_{p,q+1} \\
&\overset{(2.14)}{=} \langle \hat{v}, \hat{w} \rangle_{p,q+1} + \frac{v^0 w^0}{\sigma^2(x)} \langle \hat{x}, \hat{x} \rangle_{p,q+1} \\
&\quad - \frac{v^0}{\sigma(x)} \langle \hat{x}, \hat{w} \rangle_{p,q+1} - \frac{w^0}{\sigma(x)} \langle \hat{x}, \hat{v} \rangle_{p,q+1} \\
&\overset{(2.15)(2.16)}{=} \langle \hat{v}, \hat{w} \rangle_{p,q+1} - v^0 w^0 \\
&= \langle v, w \rangle_{p+1,q+1}.
\end{aligned}$$

■

Corollary 2.1.8. *$(S^{p,q}, g, \sigma)$ with $\sigma(x) = x^0$ is an almost Einstein structure with $S[g, \sigma] = 1$.*

Proof: The singularity set of σ and conformal boundary of the embedding is given by

$$\begin{aligned}
\Sigma &= \sigma^{-1}(0) \cap S^{p,q} \\
&= \left\{ x \in \mathbb{R}^{n+2} \;\middle|\; \begin{array}{c} (x^0 = 0), \\ \left((x^1)^2 + \cdots + (x^p)^2 = 1\right), \\ \left((x^{p+1})^2 + \cdots + (x^{n+1})^2 = 1\right) \end{array} \right\} \\
&\simeq S^{p-1} \times S^q.
\end{aligned}$$

It apparently is a $n-1$ dimensional submanifold of $S^{p,q}$. So it suffices to show that the claim holds on $S^{p,q} \setminus \Sigma$. Similar as in the proof for the embedding of the pseudo-Euclidean space one may now define a second embedding of de Sitter space into the pseudosphere by $\bar{\iota} = -\iota$. Again its inverse is given by ι^{-1} but this time defined on the image of $\bar{\iota}$. Now by defining ι^{-1} as map on the pseudosphere, one may calculate for $x \notin \Sigma$

$$\begin{aligned}
\iota \circ \iota^{-1} \left(x^0, \hat{x}\right) &= \iota \left(\frac{1}{\sigma(x)\hat{x}}\right) \\
&= \pi_{S^{n+1}_{\sqrt{2}}} \left(\frac{1}{\sigma(x)} x\right) \\
&= \operatorname{sgn}(\sigma) x.
\end{aligned}$$

Consequently ι and $\bar{\iota} = -\iota$ must cover $S^{p,q} \setminus \Sigma$ completely. As in the previous lemma, the pullback metric taken with respect to $\bar{\iota}$ still is $\sigma^{-2}\, g_{p+1,q+1}\,|_{TS^{p,q}}$. In particular the metric $\sigma^{-2} g$ is Einstein on $S^{p,q} \setminus \Sigma$ with scalar curvature $n(n-1)$. Hence $A[g, \sigma] = 0$ and Equation (1.124) gives $S[g, \sigma] = \frac{n(n-1)}{n(n-1)} = 1$. By continuity of $A[g, \sigma]$ and $S[g, \sigma]$, this then also holds on Σ. ■

Conformal Boundary

Consider the almost Einstein structure $(S^{p,q}, g, \sigma)$, arising from the embedding of the de Sitter space. The gradient of the scale factor σ with respect to g at $x \in S^{p,q}$ is

$$\operatorname{grad}_x^g \sigma = \left(-1 + \left(x^0\right)^2, x^0 x^1, \ldots, x^0 x^p, 0, \ldots, 0\right).$$

Its pseudo-norm is $\left\|\operatorname{grad}_x^g \sigma\right\|_g^2 = \left(x^0\right)^2 - 1 = \sigma^2(x) - 1$ and in particular it is -1 and timelike, where σ vanishes. Again this only confirms, what already one would have had expected by considering the definition of almost scalar curvature (Equation (1.115)). Hence Σ is a submanifold of $S^{p,q}$ of signature $(p-1, q)$.

An other important property is that no single point in the conformal boundary can be seen as asymptotic infinity of just one type of geodesics, as will be shown next. Let $\hat{x} \in dS^{p,q} \subset \mathbb{R}^{n+1}$ be a point in de Sitter space and $\hat{v} \in T_{\hat{x}} dS^{p,q}$ a tangent vector normalised such that $g(\hat{v}, \hat{v}) = g_{p,q+1}(\hat{v}, \hat{v}) \in \{-1, 0, 1\}$. In particular $\langle \hat{x}, \hat{v} \rangle_{p,q+1} = 0$. Then the geodesic $\gamma : \mathbb{R} \to dS^{p,q}$ with $\gamma(0) = x$ and initial vector $\dot{\gamma}(0) = 0$ is [O'N83, Chapter 4]

$$\gamma(t) := \begin{cases} \cos(t)\hat{x} + \sin(t)\hat{v} & g(\hat{v}, \hat{v}) = 1 \\ \hat{x} + t\hat{v} & g(\hat{v}, \hat{v}) = 0 \\ \cosh(t)\hat{x} + \sinh(t)\hat{v} & g(\hat{v}, \hat{v}) = -1. \end{cases}$$

The image of γ under the inclusion map $\imath$ is a geodesic in the pseudosphere with respect to $\sigma^{-2}g$ due to the previous considerations. Spacelike geodesics are periodic and so do not "end" at the conformal boundary. But in case where $\hat{v}$ is a causal vector, the limit $t \to \infty$ exists and can be calculated

$$\lim_{t \to \infty} \imath \circ \gamma(t) = \begin{cases} \pi_{S^{n+1}_{\sqrt{2}}}(0, \hat{v}) & g(\hat{v}, \hat{v}) = 0 \\ \pi_{S^{n+1}_{\sqrt{2}}}(0, \hat{x} + \hat{v}) & g(\hat{v}, \hat{v}) = -1. \end{cases}$$

The limit points are at the conformal boundary Σ of $\imath\left(dS^{p,q}\right)$. Each point of Σ is a limit point of both, timelike and null geodesics.

From $\Sigma \simeq S^{p-1} \times S^q$ one immediately observes that the conformal boundary decomposes into two disjoint sets if $p = 1$. The sets are distinguished by the sign of the second component of $\left(x^0, x^1, \dots\right) \in S^{1,n-1}$. If a time orientation is chosen, they may be interpreted as conformal future and conformal past of the embedded de Sitter space. The motivation comes from the following observations. On $dS^{1,n-1}$, consider the vector field, defined by $T_{\hat{x}} := e_1 + \hat{x}^1 \hat{x}$. It is tangent to $dS^{1,n-1}$ in $\mathbb{R}^{1,n}$, since $\langle T_{\hat{x}}, \hat{x} \rangle_{1,n} = 0$. It is timelike, since $\langle T_{\hat{x}}, T_{\hat{x}} \rangle_{1,n} = \langle T_{\hat{x}}, e_1 \rangle_{1,n} = -1 - \left(\hat{x}^1\right)^2 < 0$ and so defines the time orientation on $dS^{1,n-1}$. For an arbitrary vector $\hat{v} \in T_{\hat{x}} dS^{1,n-1}$ one also has $\langle T_{\hat{x}}, \hat{v} \rangle_{1,n} = \hat{v}^1$. So in particular the sign of the first component of a timelike vector defines its time orientation. Consider $\hat{v}$ to be normalised to length -1. Then it was shown that the limit point of the geodesic $\imath \circ \gamma$ is $\pi_{S^{n+1}_{\sqrt{2}}}(0, \hat{x} + \hat{v})$. The sign of its second component is the result of the following calculation. First observe

$$\begin{aligned} \left(\hat{x}^1\right)^2 \left(\hat{v}^1\right)^2 &= \left[\hat{x}^2 \hat{v}^2 + \dots + \hat{x}^{n+1} \hat{v}^{n+1}\right]^2 \\ &\leq \left[\left(\hat{x}^2\right)^2 + \dots + \left(\hat{x}^{n+1}\right)^2\right] \left[\left(\hat{v}^2\right)^2 + \dots + \left(\hat{v}^{n+1}\right)^2\right] \\ &= \left[\left(\hat{x}^1\right)^2 + 1\right] \left[\left(\hat{v}^1\right)^2 - 1\right], \end{aligned}$$

which gives $0 < \left(\hat{v}^1\right)^2 - \left(\hat{x}^1\right)^2$. Consequently the sign of $\hat{x}^1 + \hat{v}^1$ is determined by the sign of $\hat{v}^1$ and hence by the time orientation of $\hat{v}$. Finally future-directed causal curves and past-directed causal curves have limit points in different connected components of Σ, which then are named conformal future and conformal past of the embedding.

2.1.4. Anti-de Sitter Space

The third model space of dimension n, which can be conformally embedded into the pseudosphere and results in an almost Einstein structure, is the anti-de Sitter space. It has constant negative scalar curvature and its definition and its treatment coincide with those of the de Sitter space up to signs and renaming. That is why in the following only the results and basic definitions will be mentioned. Calculations then are analogous to the last section. The definition used here for the anti-de Sitter space is that of a submanifold of $\mathbb{R}^{n+1}$.

Definition 2.1.9. The *anti-de Sitter space* of signature (p,q) with $p+q=n$ is defined as embedded submanifold of $\mathbb{R}^{p+1,q}$ by

$$AdS^{p,q} = \{x \in \mathbb{R}^{n+1} \mid \langle x, x\rangle_{p+1,q} = -1\}.$$

The metric induced by the flat metric $g_{p+1,q}$ on $AdS^{p,q}$ will be denoted g^{AdS}.

$\left(AdS^{p,q}, g^{AdS}\right)$ is an Einstein manifold with negative scalar curvature $\tau^{AdS} = -\frac{1}{n(n-1)}$. A proof may be found in [O'N83, *Chapter*4]. The anti-de Sitter space can be embedded into the pseudosphere by the map

$$\begin{array}{cccc} \imath : & AdS^{p,q} & \longrightarrow & S^{p,q} \\ & (x^0, \dots, x^n) & \longmapsto & \pi_{S^{n+1}_{\sqrt{2}}}\left(x^0, \dots, x^n, 1\right). \end{array} \tag{2.17}$$

Writing $x = \left(\hat{x}, x^{n+1}\right) \in S^{p,q} \subset \mathbb{R}^{n+2}$ and defining $\sigma : \mathbb{R}^{n+2} \to \mathbb{R}$ by

$$\sigma(x) := x^{n+1} \tag{2.18}$$

then give the inverse map as restriction of $\imath^{-1} : \mathbb{R}^{n+2} \setminus \sigma^{-1}(0) \to AdS^{p,q}$ with $\imath^{-1}(x) := \frac{1}{\sigma(x)}\hat{x}$ to the image of $\imath$. By use of analogous arguments as in the previous examples one has that $\imath$ indeed is an embeddings. As a result one has the following lemma.

Lemma 2.1.10. *The pullback of $g_{p+1,q}\big|_{AdS}$ restricted to the tangent space of the anti-de Sitter space is given by*

$$\imath^{-1*} g_{p+1,q} = \sigma^{-2}\, g_{p+1,q+1}\big|_{TS^{p,q}}. \tag{2.19}$$

$(S^{p,q}, g, \sigma)$ is an almost Einstein structure with $S[g,\sigma] = -1$.

The proof of the above claim basically coincides with that given in the last subsection.

Conformal Boundary

The conformal boundary $\Sigma \simeq S^p \times S^{q-1}$ is a spacelike submanifold of the pseudosphere. Its spacelike normal vector is

$$\mathrm{grad}^g_x\, \sigma = \left(0, \dots, 0, -x^{n+1}x^{p+1}, \dots, -x^{n+1}x^n, 1 - \left(x^{n+1}\right)^2\right).$$

Geodesics in anti-de Sitter space are given by

$$\gamma(t) := \begin{cases} \cos(t)\hat{x} + \sin(t)\hat{v} & g\left(\hat{v}, \hat{v}\right) = -1 \\ \hat{x} + t\hat{v} & g\left(\hat{v}, \hat{v}\right) = 0 \\ \cosh(t)\hat{x} + \sinh(t)\hat{v} & g\left(\hat{v}, \hat{v}\right) = 1, \end{cases}$$

where $\hat{x} \in AdS^{p,q}$ is a point in anti-de Sitter space and $\hat{v} \in T_{\hat{x}}AdS$ is a normalised tangent vector. Timelike geodesics are periodic and their images $\imath \circ \gamma$ in the pseudosphere do not have a limit point at the conformal boundary. On the other hand spacelike and null geodesics have limit points $\pi_{S^{n+1}_{\sqrt{2}}}\left(\hat{x} + \hat{v}, 0\right)$ and $\pi_{S^{n+1}_{\sqrt{2}}}\left(\hat{v}, 0\right)$.

2.1.5. Visualisation of Conformal Embedding Into the Pseudosphere

The last three basic models for conformal embeddings of Einstein manifolds into the pseudosphere and the resulting almost Einstein structures on the pseudosphere can be easily visualised in the case of signature $(1,1)$. Here the pseudosphere can be identified with the torus $\mathbb{R}^3 \supset T^2 \simeq S^1 \times S^1 \simeq S^{1,1}$ and one qualitatively gets a picture as given in figure 1.

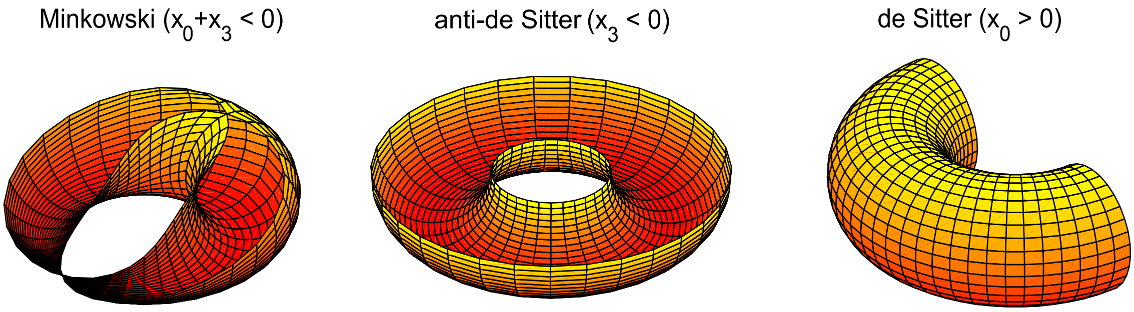

Figure 1.: Visualisation of the embedding of $\mathbb{R}^{1,1}$, $dS^{1,1}$ and $AdS^{1,1}$ into to pseudosphere. The horizontal plane carries the second S^1, which is defined by $(x^2) + (x^3)^2 = 1$, while the plane that is rotated along that circle carries the first S^1 defined by $(x^0) + (x^1)^2 = 1$. The upper critical point of $\sigma(x) = x^0 + x^3$ of the embedding of Minkowski space $\mathbb{R}^{1,1}$ corresponds to $(-1, 0, 0, 1)$, which is spacelike infinity of the given embedding. The lower critical point of σ corresponds to $(1, 0, 0, -1)$, which is timelike infinity of the given embedding. (The program Mathematica was used for the creation of the picture.)

The picture is a special case in the sense that it additionally visualises the property of conformal boundary to decompose into two disjoint sets for embeddings of $\mathbb{R}^{1,q}$, $\mathbb{R}^{p,1}$, $dS^{1,q}$ or $AdS^{p,1}$ into the pseudosphere. In the case of the pseudo-Euclidean space the decomposition is manifest after removal of the critical points $(-1, \hat{0}, 1)$ and $(1, \hat{0}, -1)$, where $\hat{0}$ is the n-dimensional zero.

2.2 POINCARÉ–EINSTEIN METRICS

A further example for almost Einstein structures are generalised *Poincaré-Einstein metrics* with conformal infinity that admit a *Fefferman-Graham expansions* at the conformal boundary.

2.2.1. Anti-de Sitter Expansion

One class of such metrics arises if one considers conformally completable Einstein metrics $\tilde{g}$ in dimension $n > 2$ with $\mathrm{Ric}[\tilde{g}] + (n-1)\tilde{g} = 0$. They appear as solutions to the following problem [FG85, FG12].

Definition 2.2.1. Let $(\Sigma, [\gamma])$ be a conformal structure of signature $(p, q-1)$ of dimension $n-1$. Let $M := \Sigma \times [0, 1)$ be a thickening with boundary $\partial M = \Sigma \times \{0\}$ and $\mathring{M} := \Sigma \times (0, 1)$ and σ the parameter on $[0, 1)$. A *anti-de Sitter like Poincaré-Einstein metric* of index (p, q) will denote a solution $\tilde{g}$ to the following problem.

(i) $\tilde{g}$ has $[\gamma]$ as conformal infinity.

(ii) $\mathrm{Ric}[\tilde{g}] + (n-1)\tilde{g}$ vanishes to infinite order in σ if n is even or 3 and to order σ^{n-3} if $n > 3$ is odd (see [FG12] for details of the order).

(iii) When written as $\tilde{g} = \sigma^{-2}\left(d\sigma^2 + g_\sigma\right)$, with g_σ being a curve of metrics on Σ, then $d\sigma^2 + g_\sigma$ is the restriction of a smooth metric h on an open set in $\Sigma \times (-1, 1)$ such that the open set and h are invariant under $\sigma \to -\sigma$.

The notation $g^{AdS} = d\sigma^2 + g^{AdS}_\sigma$ will be used for the conformal background of the Poincaré-Einstein metric in the previous problem. Construction of a formal expansion basically works as follows [Gra00, And05b, And04]. Consider a manifold of type $M = \Sigma \times [0, 1)$ and a metric of type $g = d\sigma^2 + g_\sigma$, where σ parametrises the interval $[0, 1)$ and g_σ is a smooth family of metrics on Σ. Hence σ is a geodesic defining function for $\Sigma \times \{0\}$, i.e. $\mathrm{grad}\,\sigma$ is orthonormal along $\partial M := \Sigma \times \{0\}$ and $\nabla_{\operatorname{grad}\sigma} \operatorname{grad}\sigma = 0$ (see appendix, Lemma A.1.4). Now consider Lie derivatives $\mathcal{L}^j_{\operatorname{grad}\sigma}$ to order j of g or of g_σ. Then the following notation is used

$$\begin{aligned}\dot{g}_\sigma &= \mathcal{L}_{\operatorname{grad}\sigma} g_\sigma \\ g_\sigma^{(j)} &= \mathcal{L}^j_{\operatorname{grad}\sigma} g_\sigma \\ \dot{g}_\sigma^2 &= \operatorname{tr}^{g_\sigma}_{2,3} \dot{g}_\sigma \otimes \dot{g}_\sigma .\end{aligned}$$

The trace in the right-hand side can also be taken with respect to g, but since $\operatorname{grad}\sigma$ is in the kernel of $\dot{g}$, one is left with the g_σ-trace.

On the one hand, using Equation (1.104) for the Ricci tensor of the conformally changed metric $\tilde{g} = \sigma^{-2} g$ and respecting $\|d\sigma\|_g^2 = 1$ gives

$$\sigma \operatorname{Ric}[g] = \sigma \left(\operatorname{Ric}[\tilde{g}] + (n-1)\tilde{g}\right) - (n-2)\operatorname{Hess}^g \sigma + \Delta^g \sigma g. \tag{2.20}$$

On the other hand, one has for vector fields tangent to the hypersurfaces $\Sigma_t = \Sigma \times \{t\}$ that $\operatorname{Hess}^g \sigma(X,Y) = g\left(\nabla_X \operatorname{grad}\sigma, Y\right) = -g\left(\operatorname{grad}\sigma, \nabla_X Y\right) = -g\left(\operatorname{grad}\sigma, \mathrm{I\!I}(X,Y)\right) =: -K(X,Y)$, where K is the scalar valued second fundamental form. Hence $\Delta\sigma = \operatorname{tr}^g K = H$ is the mean curvature[2] of Σ_σ, where the last notation is for level sets of σ. By contracting the Gauß equation for the splitting in a local orthonormal frame $\{e_0 = \operatorname{grad}\sigma, e_i\}$ one obtains

$$\begin{aligned}\operatorname{Ric}[g](X,Y) &= \epsilon_0 g\left(\mathrm{R}^g(X,e_0)e_0, Y\right) + \operatorname{Ric}[g_\sigma](X,Y) \\ &\quad + \epsilon_0 \sum_{i=1}^{n-1} \epsilon_i \left(-g\left(\mathrm{I\!I}(e_i,e_i), e_0\right) g\left(e_0, \mathrm{I\!I}(X,Y)\right) + g\left(\mathrm{I\!I}(X,e_i), e_0\right) g\left(\mathrm{I\!I}(e_i,Y), e_0\right)\right) \\ &= \operatorname{Ric}[g_\sigma](X,Y) - HK(X,Y) + 2\left(\operatorname{tr}^{g_\sigma}_{2,3} K \otimes K\right)(X,Y) + \left(\mathcal{L}_{\operatorname{grad}\sigma} K\right)(X,Y).\end{aligned}$$

For the last line $[X, \operatorname{grad}\sigma] = \nabla_{\operatorname{grad}\sigma} \operatorname{grad}\sigma = 0$ is used and $g(X, e_0) = 0$ for tangent vector fields. Hence $g\left(\mathrm{R}^g(X,\operatorname{grad}\sigma)\operatorname{grad}\sigma, Y\right) = \nabla_{\operatorname{grad}\sigma}\left(K(X,Y)\right) + g\left(\nabla_X \operatorname{grad}\sigma, \nabla_Y \operatorname{grad}\sigma\right) = \left(\mathcal{L}_{\operatorname{grad}\sigma} K\right)(X,Y) + \left(\operatorname{tr}^{g_\sigma}_{23} K \otimes K\right)(X,Y)$. Also for tangent vector fields, the Koszul formula provides $K(X,Y) = g\left(\nabla_X Y, \operatorname{grad}\sigma\right) = -\frac{1}{2}\left(\mathcal{L}_{\operatorname{grad}\sigma} g_\sigma\right)(X,Y)$. By assuming $\operatorname{Ric}[\tilde{g}] + (n-1)\tilde{g} = 0$ the last two formulas for the Ricci tensor then give a differential equation to the family g_σ

$$0 = \sigma \ddot{g}_\sigma - (n-2)\dot{g}_\sigma + 2H g_\sigma - \sigma\left(2\operatorname{Ric}[g_\sigma] + H\dot{g} + \dot{g}_\sigma^2\right). \tag{2.21}$$

A second equation is gained by exploiting the Codazzi equation for this foliation. Consider $\{e_i\}$ to be a local orthonormal frame with $e_0 = \operatorname{grad}\sigma$ and X a vector field that is tangent to the slices of M. Without loss of generality it is assumed that $e_{i>0}$ and X are p-synchronous. Then with the definition $K(X,Y)\operatorname{grad}\sigma = \mathrm{I\!I}(X,Y)$ one observes[3] at p that $g\left(\left(\nabla_{e_i} \mathrm{I\!I}\right)(X,e_i), \operatorname{grad}\sigma\right) = \left(\nabla^T_{e_i} K\right)(e_i, X)$ and $g\left(\left(\nabla_X \mathrm{I\!I}\right)(e_i,e_i), \operatorname{grad}\sigma\right) = \nabla_X\left(K(e_i,e_i)\right)$ for $i > 0$. Now the Codazzi equation can be applied to the terms of the Ricci tensor and one gets

$$\begin{aligned}\operatorname{Ric}[g](X, \operatorname{grad}\sigma) &= \sum_{i=1}^{n-1} \epsilon_i \left(\nabla_X\left(K(e_i,e_i)\right) - \left(\nabla_{e_i} K\right)(X, e_i)\right) \\ &= dH(X) + \left(\operatorname{div}^{g_\sigma} K\right)(X).\end{aligned}$$

$\tilde{g}$ is assumed to be Einstein, $\operatorname{Hess}^g \sigma(X, \operatorname{grad}\sigma) = g\left(\nabla_{\operatorname{grad}\sigma} \operatorname{grad}\sigma, X\right) = 0$ and $g(X, \operatorname{grad}\sigma) = 0$. Then by Equation (2.20), the left-hand side vanishes and so

$$0 = dH(X) + \left(\operatorname{div}^{g_\sigma} K\right)(X). \tag{2.22}$$

2 In contrast to the second fundamental form $\mathrm{I\!I}$ and to the mean curvature vector $\operatorname{tr}^{g_\sigma} \mathrm{I\!I}$, the scalar valued second fundamental form and the mean curvature depend on the choice of the normal vector.

3 The term $\left(\nabla_Z \mathrm{I\!I}\right)(X,Y)$ is defined as $\left(\nabla^\perp_Z\left(\mathrm{I\!I}(X,Y)\right)\right) - \mathrm{I\!I}\left(\nabla^T_Z X, Y\right) - \mathrm{I\!I}\left(X, \nabla^T_Z Y\right)$, where $\nabla^\perp_X Y$ and $\nabla^T_X Y$ are the projections of $\nabla_X Y$ to the normal and tangent component of TM at the slices. By assuming X and Y to be p-synchronous, the last two terms may be neglected at p if necessary.

Finally one can derive $\mathcal{L}_{\operatorname{grad}\sigma}K$ for tangent vector fields, which gives a Riccati equation for K

$$\begin{aligned}\left(\mathcal{L}_{\operatorname{grad}\sigma}K\right)(X,Y) &= g\left(\nabla_{\operatorname{grad}\sigma}\nabla_X Y, \operatorname{grad}\sigma\right) - g\left(\nabla_{[\operatorname{grad}\sigma,X]}Y, \operatorname{grad}\sigma\right) - g\left(\nabla_X[\operatorname{grad}\sigma,Y], \operatorname{grad}\sigma\right)\\ &= g\left(\mathrm{R}(\operatorname{grad}\sigma,X)Y, \operatorname{grad}\sigma\right) - g\left(\nabla_Y \operatorname{grad}\sigma, \nabla_X \operatorname{grad}\sigma\right).\end{aligned}$$

The last term equals $\sum_i K(X,e_i)K(e_i,Y)$. But then since $[\operatorname{grad}\sigma,X] = [\operatorname{grad}\sigma,Y] = \nabla_{\operatorname{grad}\sigma}\operatorname{grad}\sigma = 0$, the last two terms in the first line vanish and one has $\left(\mathcal{L}_{\operatorname{grad}\sigma}K\right)(X,Y) = \nabla_{\operatorname{grad}\sigma}(K(X,Y))$. Taking the trace with respect to g_σ and adding the zero term $g(\mathrm{R}(\operatorname{grad}\sigma,\operatorname{grad}\sigma)\operatorname{grad}\sigma,\operatorname{grad}\sigma) = 0$ then yield $\dot{H} = \operatorname{Ric}[g](\operatorname{grad}\sigma,\operatorname{grad}\sigma) - \|K\|^2_{g_\sigma}$. Finally using $\operatorname{Ric}[\tilde{g}] + (n-1)\tilde{g} = 0$ gives

$$0 = \sigma\dot{H} - H + \sigma\|K\|^2_{g_\sigma}. \tag{2.23}$$

The last result in particular implies $H = 0$ for the boundary surface for such metrics. Equations (2.21), (2.22) and (2.23) then provide the equations for formally calculating the Fefferman-Graham expansion for g_σ

$$g_\sigma = g_{(0)} + \sigma g_{(1)} + \sigma^2 g_{(2)} + \ldots, \tag{2.24}$$

where the coefficients are $g_{(k)} = \frac{1}{k!}g_\sigma^{(k)}\Big|_{\sigma=0}$. In case where n is even, the expansion may also include logarithmic terms of type $\sigma^k\log\sigma h_{(k)}$ for σ^{n-1} and higher orders. Requiring $H = 0$ at $\sigma = 0$, Equation (2.21) gives $g_{(1)} = 0$. By differentiating (2.21) $(j-1)$ times with respect to σ, i.e. taking its $\mathcal{L}_{\operatorname{grad}\sigma}$-Lie derivatives, the coefficients of the expansion have to satisfy

$$(j+1-n)g_\sigma^{(j)} - \operatorname{tr}^{g_\sigma} g_\sigma^{(j)} g_\sigma =|_{\sigma=0} \quad \text{terms involving derivatives } g_\sigma^{(i)} \text{ with } i<j. \tag{2.25}$$

Hence for $j < n-1$, derivatives $g_\sigma^{(j-1)}$ can in principle be calculated from lower order terms. The formal expansion that one gets by inductively differentiating of g_σ with respect to Equation (2.25) is referred to as *Fefferman-Graham expansion* [FG85]. Prescribing $g_{(0)} = g_0 = \gamma$ and following [Gra00, And04], the derivatives $g_\sigma^{(j)}\Big|_{\sigma=0}$ and hence the coefficients $g_{(j)}$ are locally determined by γ and its derivatives in the induction up to order $(n-2)$.

In particular do they vanish for odd j. This can be seen as follows. The k-th transversal Lie derivative $\mathcal{L}_{\operatorname{grad}\sigma}$ of a tensor T is denoted by $T^{(k)}$. Differentiating (2.21) k times at $\sigma = 0$ gives

$$-(n-2-k)g_\sigma^{(k+1)} + 2\left(Hg_\sigma\right)^{(k)} - k\left(2\operatorname{Ric}[g_\sigma] + H\dot{g}_\sigma + \dot{g}_\sigma^2\right)^{(k-1)} =|_{\sigma=0} 0. \tag{2.26}$$

By requiring $H = 0$ at $\sigma = 0$ one already has $g^{(1)} = 0$. An induction process then is provided by considering even k with $k < n-2$ and by assuming $g^{(j)} = 0$ for all odd $j \in \{1,3,\ldots,k-1\}$. One now uses $2H = -\operatorname{tr}^{g_\sigma}\dot{g}$ and only takes care about terms on the left-hand side that do not contain odd derivatives of g_σ of order less then $k+1$. This reduces Equation (2.26) to

$$-(n-2-k)g_\sigma^{(k+1)} - \left(\operatorname{tr}^{g_\sigma} g_\sigma^{(k+1)}\right)g_\sigma - 2k\operatorname{Ric}[g_\sigma]^{(k-1)} =|_{\sigma=0} 0. \tag{2.27}$$

If $\operatorname{Ric}[g_\sigma]^{(k-1)} = 0$ at $\sigma = 0$, then by taking the trace one obtains $\operatorname{tr}^\gamma g_0^{(k+1)} = 0$ and substituting the trace to Equation (2.27) would give $g_0^{(k+1)} = 0$.

So if the $g^{(j)}$ are supposed to vanish for odd $j < k+1$ it remains to show $\operatorname{Ric}[g_\sigma]^{(k-1)} = 0$ along Σ_0. A preliminary observation is that vanishing of odd derivatives $g^{(j)}$ up to order $k-1$ implies vanishing of odd derivatives $g^{*(j)}$ of the dual metric g^* up to order $k-1$. This can be seen as follows. First $g^{*(1)}$ is a contraction of $g^*\otimes g^{(1)}\otimes g^*$ and hence vanishes provided $g^{(1)} = 0$. All contractions will be summarised with a calligraphic $\mathcal{C}$, i.e. $g^{*(1)} = \mathcal{C}(g^*\otimes g^{(1)}\otimes g^*)$. As contractions commute with Lie derivatives this implies

$$g^{*(j)} = \sum_{|J|=j} a_J\mathcal{C}\left(g^{*(j_1)}\otimes g^{(j_2)}\otimes g^{*(j_3)}\right),$$

with multinomial coefficients a_J. Hence for $j = k-1$ odd at least one of the j_i in each term of the sum has to be odd and hence all terms have to vanish.

Now derivatives of $\mathrm{Ric}[g_\sigma]$ can be calculated. One starts by observing that $\mathrm{Ric}[g_\sigma]$ and hence $\mathrm{Ric}[g_\sigma]^{(j)}$ are horizontal, i.e. vanish if evaluated with ∂_σ in one of their arguments. So it remains to calculate $\mathrm{Ric}[g_\sigma]_p^{(k-1)}(X_p, Y_p)$ for horizontal vectors X_p and Y_p. Without loss of generality assume X_p and Y_p to be the values of horizontal lifts $X, Y \in \Gamma(ThM)$ of vector fields on Σ. Then by Remark 1.1.3 the hypersurface Ricci tensor $\mathrm{Ric}[g_\sigma](X,Y)$ can be written as $\mathrm{Ric}[g_\sigma](X,Y) = L[g^*_\sigma, \dots, g^*_\sigma, g_\sigma, \dots, \mathcal{L}_X g_\sigma, \dots,]$, where $L[A_1, \dots, A_m]$ is a linear map with values in symmetric horizontal tensors that depends only on contractions of tensor products of the A_i. The horizontal tensor g^*_σ refers to the horizontal part of g^* along Σ_σ. The Lie brackets $[\operatorname{grad}\sigma, X]$ and $[\operatorname{grad}\sigma, Y]$ vanish as X and Y are horizontal, so that on the one hand $(\mathcal{L}^j_{\operatorname{grad}\sigma}\mathrm{Ric}[g_\sigma])(X,Y) = \mathcal{L}^j_{\operatorname{grad}\sigma}(\mathrm{Ric}[g_\sigma](X,Y))$ and on the other hand $\mathcal{L}^j_{\operatorname{grad}\sigma}\mathcal{L}_X = \mathcal{L}_X\mathcal{L}^j_{\operatorname{grad}\sigma}$, $\mathcal{L}^j_{\operatorname{grad}\sigma}\mathcal{L}_X\mathcal{L}_Y = \mathcal{L}_X\mathcal{L}_Y\mathcal{L}^j_{\operatorname{grad}\sigma}$, etc. Now $\mathcal{L}_{\operatorname{grad}\sigma}$ commutes with the contraction and one gets

$$\begin{aligned}\left(\mathcal{L}^j_{\operatorname{grad}\sigma}\mathrm{Ric}[g_\sigma]\right)(X,Y) &= \mathcal{L}^j_{\operatorname{grad}\sigma}\left(L\left[g^*_\sigma, \dots, g^*_\sigma, g_\sigma, \dots, \mathcal{L}_X g, \dots,\right]\right)\\ &= \sum_{|J|=j} a_J L\left[\mathcal{L}^{j_1}_{\operatorname{grad}\sigma}g^*_\sigma, \dots, \mathcal{L}^{j_p}_{\operatorname{grad}\sigma}\mathcal{L}_X\mathcal{L}_Y g_\sigma\right]\\ &= \sum_{|J|=j} a_J L\left[g^{*(j_1)}_\sigma, \dots, \mathcal{L}_X\mathcal{L}_Y g^{(j_p)}_\sigma\right].\end{aligned}$$

Consequently $\mathrm{Ric}[g_\sigma]^{(k-1)}(X,Y)$ contains contractions of tensor products of tensors of type $g^{*(j)}$, $g^{(j)}_\sigma$, $\mathcal{L}_X g^{(j)}_\sigma$ and $\mathcal{L}_X\mathcal{L}_Y g^{(j)}_\sigma$ with $j \le k-1$. The total number $|J|$ of derivatives in each of these products is $(k-1)$ and hence odd, such that each product must contain at least one term with odd j_i. The induction process already provides that the odd derivatives of g_σ vanish along Σ_0 and so $\mathcal{L}_X\mathcal{L}_Y g^{(j)}_\sigma$ and $\mathcal{L}_X g^{(j)}_\sigma$ have to vanish as well. This finally provides vanishing of $\mathrm{Ric}[g_\sigma]^{(k-1)}$ and hence vanishing of $g^{(k+1)}$. A different approach for showing vanishing of $\mathrm{Ric}[g_\sigma]^{(k-1)}$ along Σ_0 can be found in appendix B.

The coefficients $g_{(j)}$ with j even can now directly be calculated. $g_{(2)} = \ddot{g}_0$ for example can be calculated from taking the Lie derivative $\mathcal{L}_{\operatorname{grad}\sigma}$ of (2.22) once. At $\sigma = 0$ one then has

$$-(n-3)\ddot{g}_0 - 2\left(\mathrm{Ric}[g_0] - \dot{H}g_0\right) = 0.$$

Taking the trace with respect to $g_0 = \gamma$ and having in mind that by definition $\operatorname{tr}^{g_0}\ddot{g}_0 = \operatorname{grad}\sigma\,(\operatorname{tr}^{g_0}\dot{g}) - \|\dot{g}_0\|^2_{g_0} = -2\dot{H} + 0$ then gives $2(n-2)\dot{H} - \tau[g_0] = 0$. Consequently one finds

$$g_{(2)} = \frac{1}{2}\ddot{g}_0 = -\frac{1}{n-3}\left(\mathrm{Ric}[\gamma] - \frac{1}{n-2}\gamma\right) = -2\mathrm{P}[\gamma].$$

Equations (2.22) and (2.23) then provide constraints to the choice of the coefficient $g_{(n-1)}$ [And04]. In case where n is even, it has to be transverse-traceless with respect to γ, i.e.

$$\operatorname{tr}^\gamma g_{(n-1)} = 0 \qquad\qquad \operatorname{div}^\gamma g_{(n-1)} = 0 \tag{2.28}$$

but is undetermined else. Also there appear no logarithmic terms in the formal expansion. g_σ is then formally given by

$$g_\sigma \sim \gamma + \sigma^2 g_{(2)} + \dots + \sigma^{n-2}g_{(n-2)} + \sigma^{n-1}g_{(n-1)} + \dots, \tag{2.29}$$

where higher order terms depend on γ, the choice of $g_{(n-1)}$ and its derivatives.

In case where n is odd, the choice of $g_{(n-1)}$ is constrained by

$$\operatorname{tr}^\gamma g_{(n-1)} = \omega_1 \qquad\qquad \operatorname{div}^\gamma g_{(n-1)} = \omega_2, \tag{2.30}$$

where ω_1 and ω_2 are determined by γ and its derivatives. Also the expansion contains logarithmic terms of type $\sigma^j(\log\sigma)^l$ with $j \geq n$. It is given by

$$g_\sigma \sim \gamma + \sigma^2 g_{(2)} + \cdots + \sigma^{n-3} g_{(n-3)} + \sigma^{n-1} g_{(n-1)} + \sigma^{n-1} \log\sigma h_{n-1} + \ldots, \tag{2.31}$$

where $h_{(n-1)}$ is a transverse-traceless term determined by γ and its derivatives.

By construction the conformal boundary is identified with $(\Sigma, [\gamma])$. The conformal class $[\gamma]$ is of index $(p, q-1)$. Its normal vector field $\operatorname{grad}\sigma$ is spacelike.

2.2.2. De Sitter Expansion

The initial problem of the last section can be modified in the following way

Definition 2.2.2. Let $(\Sigma, [\gamma])$ be a conformal structure of signature $(p-1, q)$ of dimension $n-1$. Let $M := \Sigma \times [0,1)$ with boundary $\partial M = \Sigma \times \{0\}$ and let $\mathring{M} = \Sigma \times (0,1)$. A *de Sitter like Poincaré-Einstein metric* of index (p,q) will denote a solution $\tilde{g}$ to the following problem.

(i) $\tilde{g}$ has $[\gamma]$ as conformal infinity.
(ii) $\operatorname{Ric}[\tilde{g}] - (n-1)\tilde{g}$ vanishes to infinite order in σ if n is even or 3 and to order σ^{n-3} if $n > 3$ is odd.
(iii) When written as $\tilde{g} = \sigma^{-2}\left(-d\sigma^2 + g_\sigma\right)$, with g_σ being a curve of metrics on Σ, then $-d\sigma^2 + g_\sigma$ is the restriction of a smooth metric h on an open set in $\Sigma \times (-1,1)$ such that the open set and h are invariant under $\sigma \to -\sigma$.

For the conformal background of a solution to this problem, the notation $g^{dS} = -d\sigma^2 + g^{dS}_\sigma$ will be used, where g^{dS}_σ indicates that the family of metrics arises from a de Sitter like problem. The construction again starts with the assumption that $g = -dt^2 + g_\sigma$ is such a metric on $M := \Sigma \times [0,1)$. $\operatorname{grad}\sigma$ now is a timelike vector field. So all terms in the previous calculations involving $\|\operatorname{grad}\sigma\|^2_g = -1$ will pick up a sign. In particular the scalar valued second fundamental form is defined by $K(X,Y)\operatorname{grad}\sigma = \mathrm{I\!I}(X,Y)$ and corresponding terms will have the opposite sign, i.e. $K(X,Y) = -g\left(\mathrm{I\!I}(X,Y), \operatorname{grad}\sigma\right) = \operatorname{Hess}^g\sigma(X,Y)$ and $g\left(\mathrm{I\!I}(X,Y), \mathrm{I\!I}(V,W)\right) = -K(X,Y)K(V,W)$. Using the Gauß equation, the Ricci tensor now has to fulfil

$$\operatorname{Ric}[g](X,Y) = \operatorname{Ric}[g_\sigma](X,Y) + HK(X,Y) - 2\left(\operatorname{tr}^{g_\sigma}_{23} K \otimes K\right)(X,Y) + \left(\mathcal{L}_{\operatorname{grad}\sigma} K\right)(X,Y).$$

Observing $2K(X,Y) = \dot{g}_\sigma$ in this signature, the curve of metrics g_σ has to fulfil the following set of equations

$$\begin{aligned}
0 &= \sigma\ddot{g}_\sigma + (n-2)\dot{g}_\sigma + 2Hg_\sigma + \sigma\left(2\operatorname{Ric}[g_\sigma] + H\dot{g} - \dot{g}^2_\sigma\right)\\
0 &= dH(X) + \left(\operatorname{div}^{g_\sigma} K\right)(X)\\
0 &= \sigma\dot{H} + H - \sigma\|K\|^2.
\end{aligned}$$

Observe that by $\mathcal{L}_{\operatorname{grad}\sigma}\sigma = d\sigma(\operatorname{grad}\sigma) = -1$ one still has to deal with the similar induction process used for anti-de Sitter structures. Again only terms of even power in σ appear in the resulting expansion up to order $n-1$. The expansion then can be formally derived in the same way as it is done in the anti-de Sitter case. M.T. Anderson pointed out that if one starts with the same boundary metric γ, the coefficients of both expansion differ only up to signs [And04].

By construction the conformal boundary is identified with $(\Sigma, [\gamma])$. The conformal class $[\gamma]$ is of index $(p-1, q)$. Moreover $\operatorname{grad}\sigma$ is a timelike normal vector field at the boundary.

2.2.3. Almost Einstein Structure Interpretation

The anti-de Sitter expansion and the analogue de Sitter expansion yields almost Einstein structure in case where the formal constructions converges to smooth metrics g^{AdS} and g^{dS}. In particular, consider (Σ, γ, Ω) to be a tuple such that (Σ, γ) is a pseudo-Riemannian manifold of signature (p, q) and dimension $n-1$ and Ω a bilinear form such that

$$\operatorname{tr}^\gamma \Omega = \begin{cases} 0 & n \text{ even} \\ \omega_1[\gamma] & n \text{ odd} \end{cases} \qquad \operatorname{div}^\gamma \Omega = \begin{cases} 0 & n \text{ even} \\ \omega_2[\gamma] & n \text{ odd,} \end{cases}$$

where ω_1 and ω_2 are the objects that constrain the term $g_{(n-1)}$ in the Fefferman-Graham expansion. Assume that the Fefferman-Graham expansion on $M = [0, \epsilon) \times \Sigma$ with initial data $g_{(0)} = \gamma$, $g_{(n-1)} = \Omega$ has a well defined, sufficiently smooth limit and denote it $g^{AdS} = d\sigma^2 + g_\sigma^{AdS}$ or $g^{dS} = -d\sigma^2 + g_\sigma^{dS}$. Then $\left(M, d\sigma^2 + g_\sigma^{AdS}, \sigma\right)$ and $\left(M, -d\sigma^2 + g_\sigma^{dS}, \sigma\right)$ are almost Einstein structures with

$$S\left[d\sigma^2 + g_\sigma^{AdS}, \sigma\right] = -1 \qquad S\left[-d\sigma^2 + g_\sigma^{dS}, \sigma\right] = 1.$$

A question of particular interest is which conformal classes $[\gamma]$ on Σ can be induced by a Poincaré-Einstein metrics with pseudo-Riemannian signature on $(0, \epsilon) \times \Sigma$? In case of Lorentzian de Sitter type Poincaré Einstein metrics this problem has been addressed in [And04, Theorem 2.1]. It states that given any real-analytic Riemannian metric γ and any real-analytic symmetric bilinear form $g_{(n-1)}$ on Σ satisfying the constraint equation for its γ-trace and γ-divergence. Then the expansion uniquely exists on a thickening $[0, \epsilon) \times \Sigma$. It in addition relates the solutions in a one to one correspondence to Riemannian anti-de Sitter solutions with the same initial data. The problem of relating the boundary metric of such a construction to pseudo-Riemannian Poincaré Einstein metrics on a thickening has been addressed in different contexts and still is an important field of research (e.g. [Kic07, FG12, And10]).

3 SINGULARITY SET OF ALMOST EINSTEIN STRUCTURES

Almost Einstein structures are a generalisation of manifolds that are conformally Einstein. This chapter will spotlight important results out of the broad field of such structures. The focus on the one hand will be on results explicitly found in terms of almost Einstein structures and on the other hand on results that have been the foundation of this thesis, in a sense that they provided concepts and ideas that are later used to analyse almost Einstein structures.

3.1 SINGULARITY SET OF ALMOST EINSTEIN STRUCTURES

This section is about results that have already been gained for almost Einstein structures and that have been the framework of this thesis. As mentioned in section 1.4, the existence of an almost Einstein structure (M, g, σ) on M is a generalisation or weakening of the demand of having an Einstein metric on M. A related point of view is that of conformal geometry. The existence of a smooth map σ on a pseudo-Riemannian manifold (M, g), such that (M, g, σ) is a non-trivial almost Einstein structure, corresponds to the existence of a non-vanishing parallel tractor in the standard tractor bundle $\mathcal{T}$ over the conformal structure $(M, [g])$. The map σ is allowed to have zeros. The possible structure of the singularity set of σ is of particular interest in this thesis. In [Gov05] a first characterisation of this set is provided. [Gov05, Proposition 2.2] states that if there is a parallel tractor on $(M, [g])$, then this corresponds to almost Einstein structures (M, g, σ) for each $g \in [g]$. The map σ can be read from the representation of the parallel tractor in the metric g. Furthermore the first jet $j^1\sigma$ of σ can vanish only at isolated points, which is equivalent to the existence of a neighbourhood of such points, where σ has non-vanishing first-order derivative. In Riemannian signature this can be complemented by non-vanishing of σ itself in a neighbourhood of such points. Obstructions to the existence of a parallel tractor are calculated in [Gov05, GN06]. The main observations concerning almost Einstein structures are summarised in the following theorems.

Theorem [Gov05, Theorem 3.1] *A pseudo-Riemannian conformal structure $(M, [g])$ with Weyl tensor W for which at each $x \in M$ the only solution $(X \lrcorner \mathrm{W})_x = 0$ is $T_xM \ni X_x = 0$ admits an Einstein metric in the conformal class if and only if there exists a non-vanishing tractor $I \in \mathcal{T}$ such that*

$$\mathrm{R}^{\mathcal{T}}(U, V)I = 0$$
$$\left(\nabla^{\mathcal{T}} \mathrm{R}^{\mathcal{T}}\right)(U, V, Z)I = 0$$

for all $U, V, Z \in \mathfrak{X}(M)$. If the requirement on the Weyl tensor is weakened in the way that the only vector field $X \in \mathfrak{X}(M)$ solving $X \lrcorner \mathrm{W} = 0$ is supposed to be the trivial vector field, then the existence of a parallel standard tractor is equivalent to the existence of a non-vanishing tractor satisfying the above conditions.

If a metric g is chosen in the conformal class, each section in the tractor bundle over M admits a representation $(\sigma, Y, \rho) \in \Gamma(L(M) \oplus TM \oplus L(M))$, where $L(M)$ is the trivial line bundle over M. The first equation then is equivalent to

$$\sigma \,\mathrm{C}^g + Y \lrcorner \mathrm{W} = 0,$$

while the second equation is equivalent to

$$\sigma^2 \mathfrak{B}^g + (n-4)\, \mathrm{W}(Y, \cdot, \cdot, Y) = 0.$$

Both equations are conformally covariant. It may be noticed here that the above equations hold on any almost Einstein structure if Y is replaced by $\mathrm{grad}^g \sigma$ (Proposition 1.4.13). In [Gov10] the results are presented in more generality and a classification of possible zero sets is given in Riemannian signature. Some of the classification results nevertheless also hold in arbitrary signature. The singularity set $\Sigma = \sigma^{-1}(0)$ can be characterised as follows.

Theorem [Gov10, Theorem 1.1 & 3.1] *Let (M, g, σ) be an almost Einstein structure with $S[g, \sigma] \neq 0$ and $\Sigma \neq \emptyset$ then Σ is a totally umbilic hypersurface.*

In [Gov10], the theorem is proven using tractor calculus. Here a proof will be sketched that avoids the usage of tractor calculus. First $S[g, \sigma] \neq 0$ implies non-vanishing of $g(\operatorname{grad}\sigma, \operatorname{grad}\sigma)$ at Σ. Hence $d\sigma$ is a non-vanishing along Σ, which makes 0 a regular value of σ. So Σ indeed is a hypersurface. One then has the additional properties of $\operatorname{grad}\sigma$ to be a vector field on Σ that is non-null, transversal, of constant length and orthogonal to Σ with respect to g. By Corollary 1.4.11 (M, g, σ) also is almost scalar constant with non-vanishing constant $N := S[g, \sigma]$. Hence using the definition of $S[g, \sigma]$, $|N|^{-\frac{1}{2}} \operatorname{grad}\sigma$ is a normal vector field on Σ. The second fundamental form in $p \in \Sigma$ for vectors $X, Y \in \Gamma(T_p\Sigma)$ is then given by

$$\begin{aligned} \mathrm{II}(X, Y) &= \frac{1}{|N|} g(\nabla_X Y, \operatorname{grad}\sigma) \operatorname{grad}\sigma \\ &= -\frac{1}{|N|} g(Y, \nabla_X \operatorname{grad}\sigma) \operatorname{grad}\sigma. \end{aligned}$$

By using vanishing of the almost Einstein tensor $A[g, \sigma]$ one gets $g(\nabla_X \operatorname{grad}\sigma, Y) = -\rho g(X, Y)$ on Σ and hence

$$\mathrm{II}(X, Y) = \frac{\rho}{|N|} g(X, Y) \operatorname{grad}\sigma.$$

The calculation doesn't depend on the choice of p and hence Σ is totally umbilic.

In case of Riemannian signature a positive almost scalar constant $S[g, \sigma] > 0$ would give the requirement $g(\operatorname{grad}\sigma, \operatorname{grad}\sigma) < 0$ on Σ. This requirement can not be fulfilled and so the singularity set is empty. The structure of the singularity set in the case of vanishing almost scalar curvature in Riemannian signature is subject to the next cited theorem. Its structure in a Lorentzian setting is part of this thesis. A result in an arbitrary signature will be given shortly. Now consider the subsets $\mathring{M}^{\pm} := \sigma^{-1}(\mathbb{R}^{\pm}) \subset M$ then in a Riemannian setting one has the following result.

Theorem [Gov10, Theorem 1.2] *Let (M, g, σ) be a Riemannian almost Einstein structure with M connected. Then σ is non-vanishing on an open dense set. In case where $S[g, \sigma] = 0$ and $\Sigma \neq \emptyset$, then Σ is a set of isolated point. These points are critical to σ. For $p \in \Sigma$ the metric $\sigma^{-2}g$ is asymptotically locally Euclidean near p and Weyl, Cotton and Bach curvatures vanish at p. In case where $S[g, \sigma] = -1$ and in addition M is closed, the sets $M \setminus \mathring{M}^-$ and $M \setminus \mathring{M}^+$ are finite unions of connected Poincaré-Einstein manifolds.*

A proof, which almost doesn't uses tractor calculus, is given next. Only for the first part the bundle $\left(\mathcal{T}, \mathbf{g}, \nabla^{\mathcal{T}}\right)$ introduced in definition 1.4.2 is used. The tuple $(\sigma, d\sigma, \rho)$ is parallel[1] with respect to $\nabla^{\mathcal{T}}$. If σ is assumed to vanish on an open set of M, then all its derivatives would have to vanish on that set. Hence $(\sigma, d\sigma, \rho)$ would have at least one zero. Since it is parallel with respect to $\nabla^{\mathcal{T}}$, it would vanish all over M, which contradicts the requirement of (g, σ) to be a non-trivial solution to $A[g, \sigma] = 0$.

In case of vanishing almost scalar curvature $S[g, \sigma] = 0$ and Riemannian signature one automatically has $d\sigma_p = 0$ for $p \in \Sigma$. By the same argument as above ρ is non-vanishing at such points and hence using the almost Einstein condition, the Hessian of σ is proportional to the metric and definite, where σ vanishes. Consequently there is a local extremum and hence $\sigma = 0$ is fulfilled only for isolated points. Moreover such points are repeller or attractors[2] of $\operatorname{grad}\sigma$. By Corollary 1.4.14 the Cotton tensor fulfils $\mathrm{C}(\operatorname{grad}\sigma, \cdot, \cdot) = 0$ and the Bach tensor satisfies $\mathfrak{B}(\operatorname{grad}\sigma, \cdot) = 0$ at any point in M. Now using $p \in \Sigma$ to be a repeller or attractor of $\operatorname{grad}\sigma$, a

1 Parallelism of $(\sigma, d\sigma, \rho)$ is part of the proof of Corollary 1.4.12
2 The proof of that property is quite similar to that of Lemma 5.1.8 and will be given there.

minor modification[3] of Proposition 1.2.6 provides $\mathfrak{B}_p = 0$ and $C_p = 0$. The Weyl tensor vanishes due to the following observation. Since one starts with an almost scalar flat manifold, one has $\|\operatorname{grad}\sigma\|_g^2 = -2\sigma\rho$. Hence in a neighbourhood of such an isolated point p, the left-hand side of $\left\| |\sigma|^{-1/2} \operatorname{grad}\sigma \right\|_g^2 = -2\rho$ is well defined and admits a smooth extension to p. The metric g is definite. So along integral curves of $\operatorname{grad}\sigma$ the quantity $|\operatorname{grad}\sigma|^{-1/2} \operatorname{grad}\sigma$ has a non-vanishing limit, as the integral curve reaches p. Using p to be a attractor or repeller, the union of all such limits in p spans the tangent space T_pM. With $W(|\operatorname{grad}\sigma|^{-1/2}\operatorname{grad}\sigma, \cdot, \cdot, \cdot) = \pm|\sigma|^{1/2} C(\cdot, \cdot, \cdot)$ one gets vanishing of the Weyl tensor at p. The property of being asymptotically locally Euclidean is a consequence of [KR00, Theorem 1.2].

Now consider the case, where $S[g, \sigma] = -1$. Then the rescaled metric $\tilde{g} = \sigma^{-2} g$ is Einstein (Equation (1.123)) with scalar curvature $\tilde{\tau} = -n(n-1)$ (Equation (1.124)). In particular $\widetilde{\mathrm{Ric}} = (n-1)\tilde{g}$ holds away from the zero set and $\tilde{g}$ is the Poincaré metric with conformal infinity induced by g. This completes the proof.

A more general treatment of the zero set Σ is achieved by the use of the Cartan geometry, which is defined by the conformal structure of an almost Einstein structure [ČGH12, ČGH14]. In the latter papers a stratification of M via the curved orbit decomposition is achieved, which also implies a decomposition of the singularity set Σ and includes results given in [Gov10] as special cases. The main result regarding Einstein structures is [ČGH14, Theorem 3.5]. For that the conformal structure $(M, [g])$ arising from an almost Einstein structure is assumed to be orientable and of signature (p, q). Then in case where $S[g, \sigma] \neq 0$, the curved orbit decomposition yields $M = M_+ \cup M_0 \cup M_-$, where $M_\pm$ are open sets and M_0 coincides with the zero set of σ. In addition to the results mentioned before, the induced Cartan geometry on M_0 is shown to be the normal Cartan geometry determined by the conformal structure $(M_0, [\gamma])$, where γ is the induced metric on M_0. In case where $S[g, \sigma] = 0$ the curved orbit decomposition yields $M = M^- \cup M_0^- \cup M_0 \cup M_0^+ \cup M^+$, where $M_0^- \cup M_0 \cup M_0^+$ coincides with the zero set of σ. If the set M_0 is non-empty it is a smoothly embedded hypersurface. The sets $M_0^\pm$ on the other hand consist of isolated points. For signature $(0, n)$ and $(n, 0)$ the set M_0 is empty, while in other signatures non-vanishing of $M_0^\pm$ leads to non-vanishing of M_0. Moreover if $M_0 \neq \emptyset$ then it locally naturally fibres over a manifold N. The fibres are of dimension one and $[g]$ induces a conformal structure of signature $(p-1, q-1)$ on N.

We will give an alternate proof to the results for $S[g, \sigma] = 0$ almost Einstein structures using standard geometric methods in Lorentzian signature. In addition we will have a closer look on the explicit structure of the zero set of σ in section 5.1 and provide another explanation for why non-vanishing of $M_0^\pm$ implies non-vanishing of M_0.

3.2 BOUNDARY REGULARITY USING THE OBSTRUCTION TENSOR

When dealing with almost Einstein structures, one may have in mind that such structures provide a method to attach a manifest infinity to an Einstein manifold that intrinsically doesn't have on. The question of uniqueness, existence and regularity of such an attachment can in some cases by answered by using Graham-Fefferman expansions. The first part of this section will deal with this method in 4 dimensions, while the second part will review a method of generalising it to higher even dimensions.

3.2.1. Boundary Regularity in 4 Dimensions

Considering Riemannian Einstein manifolds with conformal infinity in 4 dimensions, M.T. Anderson got a remarkable good understanding of the existence of Einstein metrics induced by a

3 In particular $\mathfrak{B}_p(X, \cdot)$ and $C_p(X, \cdot, \cdot)$ can be derived as limits along reparametrised integral curves γ of $\operatorname{grad}\sigma$ with $\dot{\gamma}(0) = X$ and $\gamma(0) = p$. Then $\mathfrak{B}(\dot{\gamma}, \cdot)$ and $C(\dot{\gamma}, 0, 0)$ vanish along those curves. Since this holds for arbitrary X the claim for $\mathfrak{B}_p$ and C_p follows. A more detailed discussion of that idea is given in the proof of Proposition 1.2.6.

conformal metric on the boundary. Even if the main results are gained for Riemannian signature, a couple of achievements also holds in Lorentzian signature.

The main theorem concerning conformally compact Einstein manifolds in Lorentzian signature then is the following.

Theorem [And10, Theorem 2.6] *Let Σ be a closed 3-manifold, and let (γ, κ) be a pair consisting of a real-analytic Riemannian metric γ on Σ, and a real-analytic symmetric bilinear form κ on Σ satisfying* $\operatorname{div}^{\gamma} \kappa = 0$ *and* $\operatorname{tr}_{\gamma} \kappa = 0$. *Then up to isometry there exists a unique vacuum solution to the Einstein equations* $\operatorname{Ric}[g] = 3g$ *with cosmological constant $\Lambda = 3$, which is C^{ω} conformally compact*[4]*, defined in a neighbourhood $\Sigma \times (0, \epsilon)$ of $\mathcal{I}^{+} = \Sigma \times \{0\}$, and for which the geodesic compactification $\bar{g} = t^2 g$ on $\Sigma \times [0, \epsilon)$ satisfies*

$$\bar{g} = -dt^2 + \gamma - t^2 g_{(2)} - t^3 \kappa + t^4 g_{(4)} + \dots,$$

where the g_j are uniquely determined by γ and σ.

The advantage of this theorem in comparison to results achieved by LeBrun [LeB82] and Fefferman-Graham [FG85] was to allow an arbitrary term $g_{(3)} = \kappa$ in the expansion. The regularity part of the proof makes use of the property of 4 dimensional Einstein metrics to be Bach-flat, i.e. having a vanishing Bach tensor $\mathfrak{B} = 0$ (Equation (1.43)). The corresponding equation can be written as

$$\delta d \left(\operatorname{Ric} - \frac{\tau}{6} g \right) + \mathrm{W}(\operatorname{Ric}) = 0 \tag{3.1}$$

and is conformally covariant in 4 dimensions[5]. Here δ is the L^2 adjoint to the exterior derivative d. The Ricci tensor Ric and the metric g are treated as forms with values in TM. The conformal invariance assures that if boundary regularity is found to one solution g of the Bach equation it holds for all conformal equivalent solutions and hence for all compactifications of g. Now using Equation (1.46) in 4 dimensions the above equation is equivalent to

$$\Delta^{\nabla} \operatorname{Ric} = -\frac{1}{3} \nabla^2 \tau + \frac{1}{6} \Delta \tau g + \mathcal{R} \tag{3.2}$$

where $\mathcal{R}$ contains all terms quadratic in the curvature. Comparing this formula with the corresponding formula in [And10] one realises a sign in front of $\Delta \tau g$. This originates from the usage of the positive Laplacian on functions in that paper. Using harmonic coordinates [6] now leads to the equation

$$-2\Delta^{\nabla} \operatorname{Ric}_{\alpha\beta} = \Delta \Delta g_{\alpha\beta} + (3^{rd} \text{ order terms}), \tag{3.3}$$

where $\Delta = -g^{\alpha\beta} \partial_{\alpha} \partial_{\beta}$ on the right-hand side. Without going into more detail in this summery, the Bach Equation (3.2) with certain boundary conditions then gives a non-linear hyperbolic boundary value problem with real-analytic coefficients. As a corollary a conformal compactification with given initial data (γ, κ) has the desired boundary regularity.

Existence and uniqueness then are shown by setting up the local Cauchy problem for the Bach equation in geodesic coordinates. The initial data are chosen in such a way that that they coincide with the coefficients of the Fefferman-Graham expansion of a conformally compact Einstein metric in the splitting provided by a geodesic defining coordinate t, i.e. a coordinate function t, whose gradient $\operatorname{grad}^{g} t$ has geodesics as integral curves and in addition is orthogonal to the initial surface. By applying the Gauß lemma, the metric splits as $g = -dt^2 + g_t$, where g_t corresponds to a curve of metrics on Σ with $g_0 = \gamma$. Real analyticity of the coefficients of Equation (3.2) and of the Cauchy data then via the Cauchy-Kovalevskaya theorem provides existence and uniqueness of a real analytic metric defined on $\Sigma \times [0, \epsilon)$, which solves (3.2). Then by expanding g_t in a Taylor series with respect to t it is shown that the initial data can be chosen such that this expansion coincides with the Graham-Fefferman expansion. By real analyticity of

4 C^{ω} conformally compact means that the boundary metric, which is induced by g, is real analytic.
5 Compare Equation (1.109), where the last terms vanishe due to $n - 4 = 0$
6 Harmonic coordinates are local coordinates ϕ, such that $\sum_{\mu\nu} g^{\mu\nu} \partial_{\mu} \partial_{\nu} \phi = 0$. Another interpretation is that of a harmonic gauge, in particular with vanishing wave gauge vector H and with D being the connection induced by some coordinate derivative. The corresponding coordinates then are harmonic. The harmonic gauge will be a subtopic of section 4.1

the solutions the coefficients of the Taylor series derived from the Bach equation must coincide with the coefficients of the Fefferman-Graham expansion and hence the corresponding metrics coincide. For the detailed proof we refer to the original paper [And10].

3.2.2. Boundary Regularity in Even Dimensions

The idea in the last section was to replace the Einstein equation in $n = 4$ dimensions by the Bach equation to get a better behaved system with initial data on the conformal boundary Σ. The idea was generalised to higher, even dimension by replacing the Einstein tensor with the obstruction tensor [And05a, AC05]. In [And05a] the generalisation is achieved for positive cosmological constant $\Lambda > 0$ and the initial data are prescribed at Σ. In [AC05] the focus is on Einstein metrics with $\Lambda = 0$. The initial data are given a Riemannian manifold of one dimension less, which intersects what later will be conformal infinity. A similar type of initial data has been considered in [AC96]. The focus in this section is on a short summary of the method developed for replacing the Einstein equation with a conformally covariant equation that is regular at Σ. Throughout the next section the dimension of the manifold M that is to be constructed is $n+1$. This is to simplify comparison of facts in this section with the one found in literature and may not be seen as being in disagreement with the rest of the paper. If special methods are used later, they will be reintroduced with an adapted notation. The results concerning the boundary regularity of conformally compact Einstein manifolds are formulated in terms of

H^s -	Sobolev spaces $\left(H^s := W^{s,2}\right)$ and
H^s_{loc} -	local Sobolev spaces.

Generalisation of Bach Equation to Higher Even Dimension

Let $(M, g) = \left(\tilde{M} \cup \mathcal{I}^+ \cup \mathcal{I}^-, g\right)$ be a $n+1$ dimensional manifold, and consider $\left(\tilde{M}, g\right)$ to be globally hyperbolic. The generalisation as given in [And05a] starts with the normalised Einstein equation with cosmological constant $\Lambda = \frac{n(n-1)}{2}$. It reads as $\mathrm{Ric}[\tilde{g}] = n\tilde{g}$. $\tilde{g}$ is assumed to be a solution on $\tilde{M}$ that admits a completion of the form $g = \rho^2\tilde{g}$ that also is defined at the conformal boundary $\mathcal{I}^{\pm}$. The conformal factor then gives another characterisation of the initial decomposition as $\tilde{M} = \{x \in M \,|\, \rho(x) \neq 0\}$ and $\mathcal{I}^{\pm} = \{\rho = 0\}$. One of the sets $\mathcal{I}^+$ or $\mathcal{I}^-$ may be empty or $\mathcal{I}^+$ and $\mathcal{I}^-$ may require different choices of the conformal factor ρ to give the decomposition. $\mathcal{I}^+$ then is referred to as future conformal infinity, if there are no past directed curves ending on $\mathcal{I}^+$. The analogue holds for the past conformal infinity $\mathcal{I}^-$. Globally hyperbolic Einstein manifolds $\left(\tilde{M}, \tilde{g}\right)$ with $\Lambda > 0$, admitting such a completion, are said to be asymptotically de Sitter to the future or to the past. Now as $\tilde{M}$ is globally hyperbolic, there is a spacelike Cauchy surface $\mathcal{S}$, such that $\tilde{M} \simeq S \times \mathbb{R}$. The space of such Einstein manifolds with compact Cauchy surface is denoted dS^+ or dS^-. The space of manifolds that are asymptotically de Sitter to the future and to the past is denoted by $dS^{\pm}$.

By supposing ρ to be a geodesic compactification, the Gauß lemma provides a splitting $g = -d\rho^2 + g_\rho$, where g_ρ is a curve of metrics describing the asymptotic behaviour of $\tilde{g}$ at conformal infinity as ρ goes to 0. If n is even the formal expansion of g_ρ given by Fefferman-Graham is

$$g_\rho \sim g_{(0)} + \rho^2 g_{(2)} + \cdots + \rho^{n-2} g_{(n-2)} + \rho^n g_{(n)} + \rho^n (\log \rho)\mathcal{O} + \ldots .$$

The terms $g_{(2k)}$ are determined by the boundary metric $\gamma = g_{(0)}$ up to order $n-2$. The term $g_{(n)}$ is freely choosable up to the constraints that $\mathrm{tr}_\gamma\, g_{(n)} = \alpha$ and $\delta_\gamma g_{(n)} = \beta$, where α and β are also determined by the boundary metric γ. The coefficients of higher order terms $\rho^{2k}(\log \rho)^j$ then are computable in terms of the boundary metric γ, $g_{(n)}$ and its covariant derivatives.

The term $\mathcal{O}$ is the *Fefferman-Graham obstruction tensor*. It is a symmetric, trace-free and divergence-free 2-tensor. It is determined by the boundary metric γ and a conformal covariant of the latter. Moreover its vanishing is an obstruction to the existence of a formal power series expansion of conformally compactified Einstein metrics. Although the obstruction tensor appears only as coefficient of the first logarithmic term in the Fefferman-Graham expansion for $n+1$ odd, it can be used to analyse solutions to the vacuum Einstein equation in even dimensions. In fact any solution to the Einstein equation $\mathrm{Ric} - \frac{\tau}{n+1} g = 0$ on an $n+1$ dimensional manifold with $n+1$ even also fulfils

$$\mathcal{O} = 0. \tag{3.4}$$

This property then is used in [And05a] to get boundary regularity and an existence theorem for conformally compact Einstein metrics. In contrast to the conformal Einstein equations, i.e. $A[g,\rho] = [\mathrm{Hess}\,\rho + \rho \mathrm{P}]_{\text{trace free}} = 0$, which leads to a degenerate system of equations for g on the conformal boundary due to the factor ρ in front of the Schouten tensor P, the above Equation (3.4) gives a non-degenerate system of equations in g. Assuming $n \geq 3$ odd, it is explicitly given by (cf. [GH05])

$$\mathcal{O} = \left(\Delta^{\nabla^g}\right)^{\frac{n+1}{2}-2} \left(\Delta^{\nabla^g}\mathrm{P} + \mathrm{Hess}\, J\right) + \mathcal{F}^n, \tag{3.5}$$

where J is trace of the Schouten tensor P and $\mathcal{F}^n$ is a tensor depending on derivatives of the metric up to order n. Moreover in the particular case $\mathcal{F}^n$ only depends on derivatives of the metric up to order $n-1$.

Now boundary regularity, existence and uniqueness of solutions to the equation $\mathcal{O} = 0$ can be proven by fixing certain gauges, i.e. using degrees of freedom to rewrite the differential equation such that PDE-methods can be applied. The next step is to shown that initial data can be chosen such that solutions implying them are conformally Einstein. This is important, since not all solutions to $\mathcal{O} = 0$ will be conformally Einstein. Eliminating the ambiguity in the system is done as follows. First without loosing generality one locally may choose a $\gamma \in [\gamma]$ with constant scalar curvature $\tau^\gamma = const.$. This is an application of the Yamabe problem. Next the freedom of choosing a conformally equivalent metric of g is used. Let

$$\tilde{g} = \omega^2 g$$

be the conformally changed metric. Then locally ω can be chosen, such that $\tilde{g}$ has constant scalar curvature and one may fix its explicit value by requiring $\tau^{\tilde{g}} = \tau^g|_{\mathcal{I}^+} = -\frac{n(n-2)}{n-1}\tau^\gamma$. As a consequence, Equation (3.5) simplifies to an equation on the Ricci tensor

$$0 = \left(\Delta^{\nabla^{\tilde{g}}}\right)^{\frac{n+1}{2}-1} \widetilde{\mathrm{Ric}} + \mathcal{F}^n$$

where $\mathcal{F}^n$ is a term depending on $\tilde{g}$ and its derivatives up to order n. The arbitrariness implied by the diffeomorphism invariance is exploited by choosing harmonic coordinates[7] x_α, such that $\Delta^{\nabla^{\tilde{g}}} x_\alpha = 0$. In these coordinates the Ricci curvature locally is given by

$$\widetilde{\mathrm{Ric}}_{\alpha\beta} = -\sum_{\mu,\nu} \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \partial_\nu \tilde{g}_{\alpha\beta} + Q_{\alpha\beta}\left[\tilde{g}, \partial\tilde{g}\right], \tag{3.6}$$

where $Q_{\alpha\beta}\left[\tilde{g}, \partial\tilde{g}\right]$ is quadric in $\tilde{g}$ its first derivatives. Additional in harmonic coordinates $\nabla^{\tilde{g}^*}\nabla^{\tilde{g}}$ has leading order term $\tilde{g}^{\mu\nu}\partial_\mu\partial_\nu$ and as a result Equation (3.5) reduces to a system of PDEs for $\tilde{g}_{\alpha\beta}$, diagonal in the leading order term

$$0 = \left(\tilde{g}^{\mu\nu}\partial_\mu\partial_\nu\right)^{\frac{n-1}{2}} \tilde{g}_{\alpha\beta} + \mathcal{F}^n \tag{3.7}$$

where Einstein notation is used and again $\mathcal{F}^n$ collects derivatives of $\tilde{g}$ up to order n. Now taking into account that ρ is constructed as geodesic defining function and ω arises from a conformal constant scalar curvature gauge, one gets two additional equations to the variables ρ and ω, namely

$$\partial_0 \left(\omega^2 \tilde{g}^{\alpha\beta} \partial_\alpha \rho \partial_\beta \rho\right) = 0 \tag{3.8}$$

$$\tilde{g}^{\mu\alpha}(\partial_\alpha\rho)\tilde{g}^{\nu\beta}(\partial_\beta\rho)\partial_\mu\partial_\nu\omega = L\left[D\omega, D^2\tilde{g}\right], \tag{3.9}$$

7 For a more detailed calculation of the consequences for the Laplacian and the Ricci tensor in harmonic coordinates we refer to section 4.1. It is presented there in the framework of the wave-map gauge.

where the right-hand side is a short notation for a collection of derivatives of ω up to order one and derivatives of $\tilde{g}$ up to order 2. More details on the explicit calculation are found in [And05a]. The system (3.7), (3.8) and (3.9) then can be reduced to a first-order, symmetrisable system of equations. Cauchy data to the system are arbitrary up to the constraint $\mathcal{O}\left(\operatorname{grad}^g \rho, \cdot\right) = 0$ on $\mathcal{I}^\pm$. The class of conformally Einstein metrics among the solutions depends strictly on the Cauchy data and is related to the Fefferman-Graham expansion. Cauchy data up to 1st order derivative of g, ω and ρ on $\mathcal{I}^\pm$ for conformally Einstein metrics are found to be

$$\omega = 1 \qquad \partial_0\omega = 0 \tag{3.10}$$
$$\rho = 0 \qquad \partial_0\rho = 1 \tag{3.11}$$
$$\tilde{g}_{00} = -1 \qquad \tilde{g}_{0i} = 0 \qquad \tilde{g}_{ij} = \gamma_{ij}. \tag{3.12}$$

First-order derivatives of g will have to vanish at $\mathcal{I}^\pm$. The system of differential equations (3.7) is of order $n+1$, so the Cauchy data for the metric $\tilde{g}$ then is complemented by data for derivatives $\partial_0^k \tilde{g}_{\alpha\beta}$ up to order $k = n$. The initial data can inductively be derived from lower order terms up to order n. Defining $\left(g_{(k)}\right)_{ij} := \frac{1}{k!}\partial_0^k \tilde{g}_{ij}$ with $i, j > 0$ it then remains to impose data for $g_{(0)}$ and $g_{(n)}$

$$g_{(0)} = \gamma \qquad g_{(n)} = \kappa,$$

where $\gamma \in [\gamma]$ is chosen to be of constant scalar curvature and κ is transverse traceless. Using the Fefferman-Graham expansion it then can be shown that the solution $(\tilde{g}, \rho, \omega)$ to this Cauchy problem corresponds to an Einstein manifold $\left(\tilde{M}, (\omega\rho)^{-2}\tilde{g}\right)$ in dS^+. A main theorem of the article then is

Theorem *[And05a, Theorem 1.1] The Cauchy problem for the Einstein equations with Cauchy data $(\mathcal{I}^+, [\gamma], [\kappa])$ at future conformal infinity is well-posed in $H^{s+n}(\mathcal{I}^+) \times H^s(\mathcal{I}^+)$, for any $s > \frac{n}{2} + 2$.*

Moreover the space of globally hyperbolic conformally compact Einstein manifolds with initial data on $\mathcal{I}^+$ is found to be open with respect to the $H^{n+s} \times H^n$ topology on $\mathcal{I}^+$.

Reduced Anderson-Fefferman-Graham Equations

A different approach using the idea of the last section has been described by M.T. Anderson and P. Chruściel [AC05]. There within a globally hyperbolic manifold the Cauchy data are constructed on a Riemannian hypersurface that intersects conformal infinity. The method in addition uses techniques for a hyperboloidal initial value problem as found in [AC96]. At the basis of the paper are the equations implied by the obstruction tensor to Einstein metrics. They are called Anderson-Fefferman-Graham equations. It is shown that for the Cauchy problem under consideration, they can be transformed to an equivalent auxiliary characteristic, first-order, symmetrisable hyperbolic system of equations. This is in contrast to [And05a], where pseudo differential operators are used to get the reduced system. A short introduction to the method will be given next. Also it is formulated for conformally compact Einstein metrics with vanishing cosmological term and hence vanishing scalar constant, it basically is applicable to the case where $\Lambda > 0$.

The system provided by the equation $\mathcal{O}[g] = 0$ is of order $n+1$ and Cauchy data are given by $(\mathcal{S}, \gamma, K^{(1)}, \ldots, K^{(n)})$, where $(\mathcal{S}, \gamma)$ is a n-dimensional Riemannian manifold and the $K^{(i)}$ are symmetric tensors corresponding to the t-th transversal derivative of the solution g in some Gauß coordinate system in a neighbourhood of $\mathcal{S}$. In particular writing $g = -dt^2 + \gamma(t)$ as before, one has $K^{(i)} = \frac{1}{2}\partial_t^i \gamma(t)\Big|_{t+0}$. In order to provide initial data to the equation $\mathcal{O}[g] = 0$ the set $(\gamma, K^{(1)}, \ldots, K^{(n)})$ has to fulfil the constraint equation

$$\mathcal{O}[g](e_0, \cdot) = 0,$$

where e_0 is the local vector field corresponding to the transversal derivative in Gauß coordinates $e_0 = -\operatorname{grad} t$. Under a conformal transformation the initial data themselves fulfil some transformation rule. Since the obstruction tensor is invariant with respect to conformal transformation, the class $[\gamma, K^{(1)}, \ldots, K^{(n)}]$ is considered as initial data on $\mathcal{S}$. Now using conformal constant curvature gauge, i.e. a gauge such that locally $\tau^g = 0$ and using harmonic coordinates with respect to that gauge, one gets the system of differential equations

$$\Box^{g\,\frac{n+1}{2}} g_{\alpha\beta} = -F^n_{\alpha\beta} \tag{3.13}$$

where $F^n_{\alpha\beta}$ is a smooth map that depends on x and derivatives of $g_{\alpha\beta}$ up to order n. By using Einstein notation the left-hand side is defined by $\Box^g = g^{\mu\nu}\partial_\mu\partial_\nu$. The initial data for the $g_{\alpha\beta}$ and its higher derivatives $\partial_0^i g_{\alpha\beta}$ up to order $i \leq n$ on $\mathcal{S}$ are then determined by the initial data $(\gamma, K^{(i)}, \ldots, K^{(n)})$ which one has started with and by the requirement to be in harmonic gauge. The result is the following proposition.

Proposition [AC05, Proposition 4.1] *Consider any class $(\mathcal{S}, [\gamma, K^{(i)}, \ldots, K^{(n)}])$ satisfying the constraint equations $\mathcal{O}(e_0, \cdot) = 0$ with*

$$(\gamma, K^{(i)}, \ldots, K^{(n)}) \in H^s_{loc}(\mathcal{S}) \times H^{s-1}_{loc}(\mathcal{S}) \times H^{s-n}_{loc}(\mathcal{S})$$

$s > \frac{3}{2}n + 1$, $s \in \mathbb{N}$, where $(\mathcal{S}, \gamma)$ is a Riemannian manifold and the $K^{(i)}$ are symmetric two-tensors as above. Then there exists a unique maximal globally-hyperbolic conformal Lorentzian structure $(M, [g])$ satisfying $\mathcal{O} = 0$ and a conformal embedding $\mathcal{S} \to M$, for which γ is the metric induced on $\mathcal{S}$ by g and such that $K^{(i)} = \mathcal{L}^i_{e_0} g\big|_{t=0}$ in Gaußian coordinates. Moreover the Cauchy problem with such initial data is well-posed in $H^s_{loc}(\mathcal{S}) \times H^{s-1}_{loc}(\mathcal{S}) \times H^{s-n}_{loc}(\mathcal{S})$.

The next step is to construct initial data that imply the solution to be conformally Einstein. Starting with a pair (γ, K) on $\mathcal{S}$ that fulfils the Einstein constraint equations with cosmological constant, one formally can derive in a Gaußian coordinate system

$$K^{(i)} := \frac{1}{2}\partial_t^i \gamma\Big|_{t=0}$$

The result for such initial data is that first of all they also fulfil the constraints $\mathcal{O}(e_0, \cdot) = 0$ and furthermore the globally hyperbolic solution provided by the last theorem is conformally Einstein. Initial data $(\mathcal{S}, \gamma, K)$ as constructed above are called Einstein or general relativistic data.

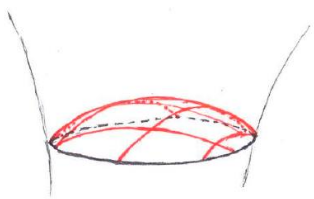

Figure 2.: schematic of the initial data set

An Einstein data set $(\mathcal{S}, \gamma, K)$ is said to have smooth conformal completion $\left(\bar{\mathcal{S}}, \bar{\gamma}, \bar{K}, \omega = \left(\omega^{(0)}, \ldots, \omega^{(n)}\right)\right)$ at infinity if the following conditions hold. $(\bar{\mathcal{S}}, \bar{\gamma})$ is conformal completion of $(\mathcal{S}, \gamma)$ with in particular $\partial\bar{\mathcal{S}} = \left(\omega^{(0)}\right)^{-1}(0)$ and $\bar{\gamma} = \left(\omega^{(0)}\right)^2 \gamma$ on $\mathcal{S}$. The collection $\omega = \left(\omega^{(0)}, \ldots, \omega^{(n)}\right)$ of smooth maps on $\bar{\mathcal{S}}$ provides the transformation of K to $\bar{K}$ under a conformal change. The first part $\omega^{(0)}$ represents the restriction of that conformal change to $\mathcal{S}$. Roughly speaking in the construction above, the $K^{(i)}$ are the i-th Lie derivatives $\mathcal{L}^i_{\partial t} g\big|_{t=0}$. So conformally changing g by a factor Ω also changes the Gaußian decomposition and Lie derivatives of the metric with respect to the new geodesic coordinate $\tilde{t}$. The new tensors $\tilde{K}^{(i)} = \mathcal{L}^i_{\partial\tilde{t}} \Omega g\Big|_{t=0}$ can be written as sum of products of derivatives of Ω and g then give a linear combination of the old $K^{(i)}$. The factors of the $K^{(i)}$ at $t = 0$ then basically are the $\omega^{(i)}$ in the above notation. Consequently $\left(\bar{\gamma} = \left(\omega^{(0)}\right)^2 \gamma, \bar{K}\right) \in [\gamma, K]$ on $\mathcal{S}$ is required to be smoothly extendible to the boundary $\partial\mathcal{S}$. It is said to be H^s if $\bar{\gamma} \in H^2(\bar{\mathcal{S}})$ and $\bar{K}^{(i)} \in H^{s-i}(\bar{\mathcal{S}})$ for $i \in \{1, \ldots, n\}$. The asymptotic behaviour then is described by the following theorem.

Theorem *[AC05, Theorem 5.2] Let $(\mathcal{S}, \gamma, K)$ be a initial data set for a $\Lambda = 0$ Einstein metric, which admits an H^s conformal completion at infinity, $s > \frac{3}{2}n + 1$, with $s \in \mathbb{N}$ and n odd. Then there exists* *an H^s Lorentzian manifold (M, g) with boundary, which is conformal completion of the unique maximal development $(\tilde{M}, \tilde{g})$ of $(\mathcal{S}, \gamma, K)$, so that $g = \sigma^2 \tilde{g}$ away from the boundary, and $\partial\mathcal{S} \subset \Sigma$*

In principle the proof can be extended to the case where $\Lambda > 0$. Cauchy data are then given at conformal infinity and the above method is another approach to the situation being subject to in [And05a].

The method used to provide existence and uniqueness of solutions to the system (3.13) with appropriate initial data roughly is the following. Consider D to be a connection on the Lorentzian manifold $\left(M^{n+1}, g\right)$. Let u be a tensor field on M and let $\Box$ be an operator with principal part $\mathrm{tr}^g D^2$. Then the system

$$\Box^{k+1} u = F\left[x, u, Du, \ldots, D^{2k+1} u\right], \tag{3.14}$$

with smooth map F is proven to be a symmetrisable hyperbolic system. It is shown [AC05, Proposition 3.1] that there exists a linear first-order operator P and a functional $H[\Phi]$, such that every solution to (3.14) with (M, g) time orientable is also a solution to the symmetrisable hyperbolic first-order system $P\Phi = H[\Phi]$. Then by construction Φ is a collection of the derivatives of u. Imposing sufficiently smooth initial data Φ, in particular $\Phi \in H^s$, existence and uniqueness of solutions are assured, if (M, g) is globally hyperbolic. The converse is shown to hold locally, given proper initial data.

4 CHARACTERISTIC CAUCHY PROBLEM FOR METRICS BEING EINSTEIN OR CONFORMALLY EINSTEIN

One has to handle a characteristic Cauchy problem, if the data are given on a surface Σ with degenerated induced bilinear form. As it will be pointed out later such challenges appear if one is interested in finding conformal structures on Σ such that they are induced by almost Einstein structures with vanishing almost scalar curvature.

This chapter deals with the challenge of posing a Cauchy problem with initial data on a null surface, such that solutions are Einstein metrics or almost Einstein structures. There are many ways of treating this issue (see for example [CP12] for a extensive survey). The focus here will be on the Cauchy problem on a null cone. First the wave-map gauge is introduced. In the context of Einstein equations or conformal Einstein equations it provides an important tool to handle or to get Laplace-type equations, if the metric is part of the unknowns. Next Friedrich's conformal field equations are presented. They represent a type of reduction of the almost Einstein equations to a first-order system. Finally the conformal wave equations, a development of Friedrich's conformal field equations, introduced in [Pae13] in dimension 4 are presented in an index-free notation. This last part of the chapter is an attempt to abstract from the ideas in [Pae13] in such a way that it may admit a generalisation of the conformal wave equations to higher even dimensions.

4.1 WAVE-MAP GAUGE

The wave-map gauge is a method that generalises harmonic coordinates. A metric g on a manifold M is said to be in $\hat{g}$-wave-map gauge if the identity map $\mathrm{id} : (M, g) \to (M, \hat{g})$ is a harmonic diffeomorphism. Definitions that are required to introduce such a gauge and its consequences to selected operators and the Einstein equation are the subject of this section. The term "wave" refers to the origin in PDEs of Laplace type $\Delta^g u = f$ in Lorentzian signature, where f may depend on x, u and derivatives of u. In that case the operator Δ^g often is written as $\Box^g$. Harmonic coordinates fulfil such an equation namely $\Delta^g x^i = 0$ or alternatively $\Box^g x^i = 0$.

Consider pseudo-Riemannian manifolds (M, g) and (N, h) and a map $\varphi : (M, g) \to (N, h)$. The following conventions are used throughout the section. The pullback $\varphi^* h$ of the metric h is called *first fundamental form* of φ. The differential $d\varphi$ is considered to be a section of the bundle $T^*M \otimes \varphi^* TN$ over M, with $(\varphi^* X)_x = X_{\varphi(x)}$. The covariant derivative ∇ on $T^*M \otimes \varphi^* TN$ is the generic connection on tensor products acting on the first part as Levi Civita connection ∇^g and on the second part as the pullback Levi Civita connection of ∇^h. The pullback connection $\varphi^* \nabla^h$ is uniquely defined by requiring $\left(\varphi^* \nabla^h\right)_X (\varphi^* Y) = \varphi^* \left(\nabla^h_{d\varphi(X)} Y\right)$ on pullback sections. Consequently $\nabla(d\varphi)$ can be treated as section of $T^*M \otimes T^*M \otimes \varphi^* TN$. Now $\nabla(d\varphi)$ is called *second fundamental form* of φ.

Definition 4.1.1. With the above conventions the metric trace of the second fundamental form of a smooth map $\varphi : (M, g) \to (N, h)$

$$\kappa(\varphi) := \mathrm{tr}^g_{12} \nabla(d\varphi)$$

is called the *tension field* of φ. φ is said to be *harmonic*, if it has vanishing tension $\kappa(\varphi) = 0$.

Definition 4.1.2. Let g and $\hat{g}$ be two pseudo-Riemannian metrics on M and ∇^g or $\nabla^{\hat{g}}$ be the corresponding Levi-Civita connections. Consider the $(2, 1)$-potential $\mathcal{M}$ defined by $\mathcal{M}(X, Y) :=$

$\nabla^g_X Y - \nabla^{\hat{g}}_X Y$ and let $\{e_i\}$ to be an local orthonormal frame with respect to g. Then g is said to be in *$\hat{g}$-wave-map gauge* if the identity map id : $(M, g) \to (M, \hat{g})$ is a harmonic diffeomorphism. The vector field

$$H := \operatorname{tr}^g_{1,2} \mathcal{M} = \sum_i \epsilon_i \left(\nabla^g_{e_i} e_i - \nabla^{\hat{g}}_{e_i} e_i \right)$$

is called the *wave-gauge vector*.

Remark. The tension of the identity map id : $(M, g) \to (M, \hat{g})$ is given by the wave-map gauge vector with respect to g and $\hat{g}$,

$$\kappa(\mathrm{id}) = -H.$$

Consequently g is in $\hat{g}$ wave-map gauge if the associated wave-gauge vector vanishes.

Proof: The differential of the identity map is the identity map on the tangent space TM. Therefore choosing a local frame $\{e_i\}$, orthonormal with respect to g and writing $\{\sigma^i\}$ for its dual frame, locally $d\,\mathrm{id}$ can be written as

$$d\,\mathrm{id} = \sum_j \sigma^j \otimes e_j.$$

The covariant derivative of $\sigma^j \otimes e_j$ in the direction of e_i evaluated with e_k is

$$\begin{aligned}
\left(\nabla_{e_i} \sigma^j \otimes e_j \right)(e_k) &= \left((\nabla^g_{e_i} \sigma^j) \otimes e_j + \sigma^j \otimes \nabla^{\hat{g}}_{e_i} e_j \right)(e_k) \\
&= \left(\nabla^g_{e_i} \sigma^j \right)(e_k) e_j + \sigma^j(e_k) \nabla^{\hat{g}}_{e_i} e_j \\
&= \left(\nabla^g_{e_i} \left(\sigma^j(e_k) \right) \right) e_j - \sigma^j \left(\nabla^g_{e_i} e_k \right) e_j + \delta^j_k \nabla^{\hat{g}}_{e_i} e_j \\
&= 0 - \sigma^j \left(\nabla^g_{e_i} e_k \right) e_j + \delta^j_k \nabla^{\hat{g}}_{e_i} e_j.
\end{aligned}$$

By taking the metric trace with respect to g one obtains

$$\begin{aligned}
\operatorname{tr}^g_{12} d\,\mathrm{id} &= \sum_{i,j,k} g(e_i, e_k) \left(\nabla_{e_i} \sigma^j \otimes e_j \right)(e_k) \\
&= -\sum_{i,j,k} g(e_i, e_k) \sigma^j \left(\nabla^g_{e_i} e_k \right) e_j + \sum_{i,j,k} g(e_i, e_k) \delta^j_k \nabla^{\hat{g}}_{e_i} e_j \\
&= -\sum_{i,k} g(e_i, e_k) \sum_j \sigma^j \left(\nabla^g_{e_i} e_k \right) e_j + \sum_{i,j} g(e_i, e_j) \nabla^{\hat{g}}_{e_i} e_j \\
&= -\sum_{i,k} g(e_i, e_k) \left(\nabla^g_{e_i} e_k - \nabla^{\hat{g}}_{e_i} e_k \right) \\
&= -H.
\end{aligned}$$

■

4.1.1. Reduced Ricci Tensor

Denote the Levi-Civita connection of $\hat{g}$ with D. Then using the wave-gauge vector, the Ricci tensor has a decomposition $\mathrm{Ric} = \mathrm{Ric}^{(H)} + F[H]$, where $\mathrm{Ric}^{(H)}$ is a term that only depends on second derivatives of the metric of type $\left(\operatorname{tr}^g D^2\right) g$. The last term involves second derivatives of the metric only as part of the wave-gauge vector. This decomposition is valuable, if the Ricci tensor is considered as source for a PDE, for which also the metric is considered to be an unknown. To provide this well known decomposition and to give a mostly index-free calculation to get it is aim of the this subsection.

Lemma 4.1.3. *Let g and $\hat{g}$ be pseudo-Riemannian metrics on M and let $\nabla := \nabla^g$ or $D := \nabla^{\hat{g}}$ be the associated Levi-Civita connections. Then the following decomposition holds for the Ricci tensor of g*

$$\mathrm{Ric}(X, Y) = \mathrm{Ric}^H(X, Y) + \frac{1}{2} \left(g(X, D_Y H) + g(Y, D_X H) \right). \tag{4.1}$$

where

$$\mathrm{Ric}^{(H)} = -\frac{1}{2}\,\mathrm{tr}^g\, DDg + Q[g, Dg] \tag{4.2}$$

is called the reduced Ricci tensor. Q is a tensor, which is a quadric in Dg and has coefficients depending only on the metrics g and $\hat{g}$. In particular it is independent of second derivatives of g.

The advantage of this decomposition emerge if $\hat{g}$ can be chosen such that the wave-gauge vector vanishes. If considering Cauchy type problems it is easier to deal with the reduced Ricci tensor $\mathrm{Ric}^{(H)}$ than with the full Ricci tensor. The decomposition can be obtained as follows. First denote by $\mathcal{M}$ the *potential* of ∇ with respect to D, i.e.

$$\mathcal{M}(X, Y) := (\nabla_X - D_X)\, Y. \tag{4.3}$$

Then the gauge-wave vector of g and $\hat{g}$ is the trace $H = \mathrm{tr}^g\, \mathcal{M}$. Now neglecting for a moment the fact that the connections ∇ and D are Levi-Civita connections one has the following well known more general properties. Consider ∇ and D to be torsion-free connections, then the potential $\mathcal{M}$ as defined in Equation (4.3) is symmetric, i.e. $\mathcal{M}(X, Y) = \mathcal{M}(Y, X)$. This is a consequence of $\nabla_X Y - \nabla_Y X = [X, Y] = D_X Y - D_Y X$. Then in particular one has

Lemma 4.1.4. *Let ∇ and D be two torsion-free connections on M and $\mathcal{M}$ the potential as defined above. Assume both connections to be canonically extended to act on arbitrary tensor fields T by Equation (1.1). Let X be a vector field on M. Then the covariant derivative of T can be expressed by using the Ricci product (1.20), as*

$$\nabla_X T = D_X T + \mathcal{M}(X) \cdot T, \tag{4.4}$$

where $\mathcal{M}(X) := \mathcal{M}(X, \cdot)$ is a $(1,1)$-tensor field.

A short proof is provided in the appendix. To calculate higher order derivatives of T, the Ricci product notation is modified on the right-hand side of Equation (4.4) by

$$(\mathcal{M} \cdot T)(X, \cdot, \ldots, \cdot) := (\mathcal{M}(X) \cdot T)(\cdot, \ldots, \cdot).$$

This will help to keep calculations short. Also a Leibniz rule can be formulated. Let D be a torsion-free connection on arbitrary tensor fields that is a generic extension of a connection on TM then it holds

$$(D(\mathcal{M} \cdot T))\,(Y, X, \cdot, \ldots, \cdot) = ((D_Y \mathcal{M})(X) \cdot T)\,(\cdot, \ldots, \cdot) + (\mathcal{M}(X) \cdot DT)\,(Y, \cdot, \ldots, \cdot). \tag{4.5}$$

Lemma 4.1.5. *Let (M, g) be a pseudo-Riemannian manifold with Levi-Civita connection ∇ and assume D to be a torsion-free connection, which differs from ∇ by the potential $\mathcal{M}$, i.e. $\nabla - D = \mathcal{M}$. Then for $p \in M$ and $X, Y, Z \in T_pM$ it holds*

$$\begin{array}{rrcl} (i) & (Dg)(X,Y,Z) & = & g(\mathcal{M}(X,Z),Y) + g(\mathcal{M}(X,Y),Z) \\ (ii) & 2g\,(Z, \mathcal{M}(X,Y)) & = & (Dg)(Y,Z,X) + (Dg)(X,Y,Z) - (Dg)(Z,X,Y) \\ (iii) & \mathrm{R}^{\nabla}(X,Y)Z & = & \mathrm{R}^{D}(X,Y)Z + (D\mathcal{M})(X,Y,Z) - (D\mathcal{M})(Y,X,Z) \\ & & & +\mathcal{M}(X, \mathcal{M}(Y,Z)) - \mathcal{M}(Y, \mathcal{M}(X,Z)) \end{array} \tag{4.6}$$

The second formula expresses the property of $\mathcal{M}$ only to depend on first derivatives Dg of the metric, while contraction of the third formula will be used to decompose the Ricci tensor later.
Proof: Using Lemma 4.1.4 and that ∇ is compatible with the metric gives

$$D_X g = 0 - \mathcal{M}(X) \cdot g$$

and the claim follows by definition of the Ricci product (1.20). For the second case one first uses that ∇ and D have vanishing torsion and the result of (i) to get

$$2g\,(Z, \mathcal{M}(X,Y)) = g\,(Z, \mathcal{M}(X,Y)) + g\,(Z, \mathcal{M}(Y,X))$$

$$
\begin{aligned}
&= g(Z, \mathcal{M}(X,Y)) + g(Y, \mathcal{M}(X,Z)) \\
&\quad + g(Z, \mathcal{M}(Y,X)) + g(X, \mathcal{M}(Y,Z)) \\
&\quad - g(X, \mathcal{M}(Y,Z)) - g(Y, \mathcal{M}(X,Z)) \\
&= (Dg)(Y,Z,X) + (Dg)(X,Y,Z) - (Dg)(Z,X,Y)
\end{aligned}
$$

For the third part one has to derive $(\nabla\nabla Z)(X,Y)$ in terms of D and $\mathcal{M}$. For that one uses the definition

$$(\nabla\nabla Z)(X,Y) = \nabla_X(\nabla_Y Z) - (\nabla Z)(\nabla_X Y)$$

and replaces every appearance of ∇ with $D + \mathcal{M}$. The result is

$$
\begin{aligned}
(\nabla\nabla Z)(X,Y) = &(DDZ)(X,Y) + (D\mathcal{M})(X,Y,Z) \\
&+ \mathcal{M}(Y, D_X Z) + \mathcal{M}(X, D_Y Z) + \mathcal{M}(X, \mathcal{M}(Y,Z)) \\
&- (DZ)(\mathcal{M}(X,Y)) - \mathcal{M}(\mathcal{M}(X,Y), Z)\,.
\end{aligned} \tag{4.7}
$$

Now the curvature tensor $\mathrm{R}^{\nabla}(X,Y)Z = (\nabla\nabla Z)(X,Y) - (\nabla\nabla Z)(Y,X)$ of ∇ can be rewritten in terms of R^D, the potential $\mathcal{M}$ and the derivative D as stated. ∎

Corollary 4.1.6. *Let $H = \mathrm{tr}^g \mathcal{M}$ be the trace of the potential in the last lemma. Consider X, Y, e_i to be vector fields that are p-synchronous with respect to D. Let in addition $\left\{(e_i)_p\right\}$ be a orthonormal basis of T_pM, then at p one obtains $\sum_i g((D\mathcal{M})(X, e_i, e_i), Y) = g(D_X H, Y)$. Contraction of Equation (4.6)(iii) then gives*

$$
\begin{aligned}
\mathrm{Ric}(X,Y) = \mathrm{Ric}^{g,D}(X,Y) &+ g(D_X H, Y) + g(\mathcal{M}(X,H), Y) \\
&- \sum_m \epsilon_m g((D\mathcal{M})(e_m, X, e_m), Y) - \sum_m \epsilon_m g(\mathcal{M}(e_m, \mathcal{M}(X, e_m)), Y)
\end{aligned} \tag{4.8}
$$

where

$$\mathrm{Ric}^{g,D}(X,Y) := \sum_m \epsilon_m g(\mathrm{R}^D(X, e_m)e_m, Y) = \mathrm{tr}^g_{23}\left(\mathrm{R}^D\right)^\flat(X,Y) \tag{4.9}$$

will be called generalised Ricci tensor. This is different from the term usually being referred to as generalised Ricci tensor, since this definition contains a metric trace and therefore depends on the choice of metric.

Lemma 4.1.7. *Consider $X, Y, A \in \Gamma(TM)$ and ∇, D and $\mathcal{M}$ as before then it holds*

$$
\begin{aligned}
(i)\quad & g((D\mathcal{M})(X,A,A), Y) + g((D\mathcal{M})(Y,A,A), X) = \\
&\quad g(Y, D_X(\mathcal{M}(A,A))) + g(X, D_Y(\mathcal{M}(A,A))) \\
&\quad - 2g(Y, \mathcal{M}(D_X A, A)) - 2g(X, \mathcal{M}(D_Y A, A))
\end{aligned} \tag{4.10}
$$

$$
\begin{aligned}
(ii)\quad & g((D\mathcal{M})(A,X,A), Y) + g((D\mathcal{M})(A,Y,A), X) = \\
&\quad (DDg)(A,A,Y,X) - (Dg)(A, \mathcal{M}(X,A), Y) - (Dg)(A, \mathcal{M}(Y,A), X)
\end{aligned} \tag{4.11}
$$

Proof: The proof uses several times the properties of the last lemma and the Leibniz rule for connections. The fist equation is a direct consequence of the Leibniz rule in connection with the symmetry of $\mathcal{M}$.

For the second equation one calculates

$$
\begin{aligned}
2g((D\mathcal{M})(A,X,A), Y) &= 2g(D_A(\mathcal{M}(X,A)), Y) - 2g(\mathcal{M}(D_A X, A), Y) - 2g(\mathcal{M}(X, D_A A), Y) \\
&= 2D_A(g(\mathcal{M}(X,A), Y)) - 2(D_A g)(\mathcal{M}(X,A), Y) - 2g(M(X,A), D_A Y) \\
&\quad - 2g(\mathcal{M}(D_A X, A), Y) - 2g(\mathcal{M}(X, D_A A), Y) \\
&\overset{4.1.5(ii)}{=} D_A((Dg)(A,Y,X)) + D_A((Dg)(X,A,Y)) - D_A((Dg)(Y,A,X)) \\
&\quad - 2(D_A g)(\mathcal{M}(X,A), Y) - 2g(M(X,A), D_A Y) \\
&\quad - 2g(\mathcal{M}(D_A X, A), Y) - 2g(\mathcal{M}(X, D_A A), Y)
\end{aligned}
$$

$$
\begin{aligned}
&= (DDg)(A,A,Y,X) \\
&\quad + (Dg)(D_A A, Y, X) + (Dg)(A, D_A Y, X) + (Dg)(A, Y, D_A X) \\
&\quad + D_A\left((Dg)(X,A,Y)\right) - D_A\left((Dg)(Y,A,X)\right) \\
&\quad - 2(D_A g)\left(\mathcal{M}(X,A),Y\right) - 2g\left(M(X,A), D_A Y\right) \\
&\quad - 2g\left(\mathcal{M}\left(D_A X, A\right), Y\right) - 2g\left(\mathcal{M}\left(X, D_A A\right), Y\right)
\end{aligned}
$$

As one can see in the last part of the equation, the second line already is symmetric in X and Y, while the third line will vanish if symmetrised in X and Y. If symmetrised the last two lines cancel by using Lemma 4.1.5(ii). Hence one gets

$$
\begin{aligned}
g\left((D\mathcal{M})(A,X,A),Y\right) + g\left((D\mathcal{M})(A,Y,A),X\right) = &\\
(DDg)(A,A,X,Y)&\\
+ (Dg)(D_A A, X, Y) + (Dg)(A, D_A Y, X) + (Dg)(A, Y, D_A X)&\\
- (Dg)(A, M(X,A), Y) - (Dg)(A, M(Y,A), X)&\\
- (Dg)(D_A A, X, Y) - (Dg)(A, D_A Y, X) - (Dg)(A, Y, D_A X)&
\end{aligned}
$$

and the second claim follows immediately. ■

The splitting of the Ricci tensor in Lemma 4.1.3 then follows by making the symmetry of the Ricci tensor explicit. Let X, Y vector fields and let $\{(e_i)_p\}$ be a local orthonormal frame. Then

$$
\begin{aligned}
2\,\mathrm{Ric}^g(X,Y) &= \sum_m \epsilon_m g\left(\mathrm{R}^\nabla(X,e_m)e_m, Y\right) + g\left(\mathrm{R}^\nabla(Y,e_m)e_m, X\right) \\
&\overset{(4.8)}{=} g(D_X H, Y) + g(D_Y H, X) \\
&\quad + \mathrm{Ric}^{g,D}(X,Y) + \mathrm{Ric}^{g,D}(Y,X) + g\left(\mathcal{M}(X,H),Y\right) + g\left(\mathcal{M}(Y,H),X\right) \\
&\quad - \sum_m \epsilon_m \left(g\left((D\mathcal{M})(e_m,X,e_m),Y\right) + g\left((D\mathcal{M})(e_m,Y,e_m),X\right)\right) \\
&\quad - \sum_m \epsilon_m \left(g\left(\mathcal{M}(e_m,\mathcal{M}(X,e_m)),Y\right) + g\left(\mathcal{M}(e_m,\mathcal{M}(Y,e_m)),X\right)\right) \\
&\overset{(4.11),(4.6)(i)}{=} -(\mathrm{tr}^g_{1,2}\, DDg)(X,Y) + g(D_X H, Y) + g(D_Y H, X) \\
&\quad + \mathrm{Ric}^{g,D}(X,Y) + \mathrm{Ric}^{g,D}(Y,X) + (Dg)(H,X,Y) \\
&\quad + \sum_m \epsilon_m \left((Dg)\left(e_m,\mathcal{M}(X,e_m),Y\right) + (Dg)\left(e_m,\mathcal{M}(Y,e_m),X\right)\right) \\
&\quad - \sum_m \epsilon_m \left(g\left(\mathcal{M}(e_m,\mathcal{M}(X,e_m)),Y\right) + g\left(\mathcal{M}(e_m,\mathcal{M}(Y,e_m)),X\right)\right) \\
&\overset{(4.6)(i)}{=} -(\mathrm{tr}^g_{1,2}\, DDg)(X,Y) + g(D_X H, Y) + g(D_Y H, X) \\
&\quad + \mathrm{Ric}^{g,D}(X,Y) + \mathrm{Ric}^{g,D}(Y,X) + (Dg)(H,X,Y) \\
&\quad + 2\sum_m \epsilon_m g\left(\mathcal{M}(e_m,X),\mathcal{M}(Y,e_m)\right)
\end{aligned}
$$

Finally this leads to the proof of Lemma 4.1.3 and an explicit form of $Q[g,Dg]$ in Equation (4.2)

$$
2Q[g,Dg](X,Y) = \mathrm{Ric}^{g,D}(X,Y) + \mathrm{Ric}^{g,D}(Y,X) + (Dg)(H,X,Y) + 2\sum_m \epsilon_m g\left(\mathcal{M}(e_m,X),\mathcal{M}(Y,e_m)\right).
$$

To make it evident that Q only depends on g and DF one may use Lemma 4.1.5 to express the potential $\mathcal{M}$ in terms of g and Dg.

4.1.2. Laplace-Type Operators

Let (M,g) be a pseudo-Riemannian manifold and ∇ its Levi-Civita connection. Consider an additional torsion-free connection D on TM, extended in the generic way to a connection on $T^{p,q}M$ by Equation (1.1).

Definition 4.1.8. The *Laplace-Beltrami operator* Δ^∇ of the Levi-Civita connection on tensor fields $T \in \mathcal{T}^{p,q}M$ locally is defined by

$$
(\Delta^\nabla T)(X_1,\dots,X_p,\omega_1,\dots,\omega_q) := -\sum_i \epsilon_i (\nabla\nabla T)\left(e_i,e_i,X_1,\dots,X_p,\omega_1,\dots,\omega_q\right) \tag{4.12}
$$

where $X_j \in \mathfrak{X}(U \subset M)$ and $\omega_j \in \Omega(U \subset M)$ are local sections and $\{e_i\}$ is a local orthonormal frame. The operator $\Delta^{g,D}$ is defined by a similar equation

$$(\Delta^{g,D} T)(\dots) := -\sum_i \epsilon_i (DDT)(e_i, e_i, \dots). \tag{4.13}$$

If g has Lorentzian signature, the first Laplacian also is denoted $\Box^g := \Delta^\nabla$ and it is called *d'Alembert operator* or *wave operator*.

The aim of this section is to characterise the difference of those two Laplacians. The calculation in case where D is locally defined by coordinate derivatives is given in [Pae13].

Lemma 4.1.9. *Let $X \in \Gamma(M)$ be a vector field on (M, g), $\{e_i\}$ local orthonormal frames and the Laplace operators Δ^∇ and $\Delta^{g,D}$ defined as above. Then it holds*

$$\Delta^\nabla X - \Delta^{g,D} X = \mathrm{Ric}^\sharp(X) - \left(\mathrm{Ric}^{g,D}\right)^\sharp(X) + [H, X] - 2\sum_i \epsilon_i \mathcal{M}(e_i, (DX)(e_i)),$$

where $g\left(\left(\mathrm{Ric}^{g,D}\right)^\sharp(X), Y\right) := \mathrm{Ric}^{g,D}(X, Y)$.

Proof: Let be $p \in M$ and consider Y, e_i to be local vector fields that are p-synchronous with respect to D and assume $\{e_i\}$ to be orthonormal in p. Using Equation (4.7) and symmetry of the second term therein at p one finds

$$\begin{aligned} g(Y, \Delta^\nabla X) = & -\sum_i \epsilon g(Y, (\nabla\nabla X)(e_i, e_i)) \\ = & -\sum_i \epsilon_i g(Y, (DDX)(e_i, e_i)) - \sum_i \epsilon_i g(Y, (D\mathcal{M})(e_i, X, e_i)) \\ & + g(Y, D_H X) - 2\sum_i \epsilon_i g(Y, \mathcal{M}(e_i, (DX)(e_i))) \\ & + g(Y, \mathcal{M}(H, X)) - \sum_i \epsilon_i g(Y, \mathcal{M}(e_i, \mathcal{M}(e_i, X))). \end{aligned}$$

The second term can be replaced using Equation (4.8) to be left with

$$\begin{aligned} g(Y, \Delta^\nabla X) = & \, g(Y, \Delta^{g,D} X) + \mathrm{Ric}(X, Y) - \mathrm{Ric}^{g,D}(X, Y) - g(Y, D_X H) - g(Y, \mathcal{M}(X, H)) \\ & + \sum_i \epsilon_i g(Y, \mathcal{M}(e_i, \mathcal{M}(e_i, X))) - \sum_i \epsilon_i g(Y, \mathcal{M}(e_i, \mathcal{M}(e_i, X))) \\ & + g(Y, D_H X) - 2\sum_i \epsilon_i g(Y, \mathcal{M}(e_i, (DX)(e_i))) + g(Y, \mathcal{M}(H, X)). \\ = & \, g(Y, \Delta^{g,D} X) + \mathrm{Ric}(X, Y) - \mathrm{Ric}^{g,D}(X, Y) + g(Y, [H, X]) \\ & - 2\sum_i \epsilon_i g(Y, \mathcal{M}(e_i, (DX)(e_i))). \end{aligned}$$

The claim follows immediately. ■

A formula for the difference of the two Laplacians Δ^∇ and $\Delta^{g,D}$ can be calculated for arbitrary tensors. Explicitly it will be given only on forms, functions and tensor products of forms and vector fields.

Lemma 4.1.10. *With the requirements of the last lemma, let $f \in C^\infty(M)$ be a smooth map, then*

$$\Delta^\nabla f - \Delta^{g,D} f = df(H). \tag{4.14}$$

For forms $\omega \in \Omega(M)$ the difference of the two Laplacians is given by

$$\begin{aligned} (\Delta^\nabla \omega)(X) - (\Delta^{g,D} \omega)(X) = & -\mathrm{Ric}(X, \omega^\sharp) + \mathrm{Ric}^{g,D}(X, \omega^\sharp) + \omega(D_X H) + (D\omega)(H, X) \\ & + 2\sum_i \epsilon_i (D\omega)(e_i, \mathcal{M}(e_i, X)) - 2\sum_i \epsilon_i \omega(\mathcal{M}(e_i, \mathcal{M}(e_i, X))). \end{aligned} \tag{4.15}$$

For tensor products $T = T_1 \otimes \cdots \otimes T_p \otimes T_{p+1} \otimes \cdots T_{p+q} \in \mathcal{T}^{p,q}M$ of basic tensors $T_1, \ldots, T_p \in \Omega(M)$ and $T_{p+1}, \ldots, T_{p+q} \in \mathfrak{X}(M)$ the Leibniz rule then gives

$$\begin{aligned}\Delta^{\nabla} T - \Delta^{g,D} T = &\sum_i T_1, \otimes \cdots \otimes \left(\Delta^{\nabla} T_i - \Delta^{g,D} T_i\right) \otimes \cdots \otimes T_{p+q} \\ &+ 2 \sum_{\substack{i<j \\ k}} \epsilon_k T_1, \otimes \cdots \otimes (\mathcal{M}(e_k, \cdot) \cdot T_i) \otimes \cdots \otimes \left(\mathcal{M}(e_k, \cdot) \cdot T_j\right) \otimes \cdots \otimes T_{p+q},\end{aligned} \tag{4.16}$$

where $\mathcal{M}(e_i, \cdot) \cdot T_j$ is the Ricci product of the $(1,1)$-tensor $\mathcal{M}(e_i, \cdot)$ and T_j.

If in the last equation the differences $\left(\Delta^{\nabla} T_i - \Delta^{g,D} T_i\right)$ on the right-hand side are replaced by the formulas given for vector fields and forms, it is clear that the left-hand side is a first-order operator of D on T. Moreover the form as it is written down simplifies the identification of terms that contain second derivatives of the metric with respect to the connection D. Namely one term is DH, since H depends on first derivatives of g by Lemma 4.1.5 and the other term is the Ricci tensor, since it contains derivatives of H and $\mathcal{M}$ by Equation (4.8).
Proof: Let $\{e_i\}$ be a local orthonormal frame. The first Equation (4.14) is a consequence of the fact that on functions the connections D and ∇ acts as exterior derivative $\nabla f = df = Df$ and so

$$\begin{aligned}\Delta^{\nabla} f &= -\sum_i \epsilon_i (\nabla df)(e_i, e_i) \\ &= -\sum_i \epsilon_i \left((Ddf)(e_i, e_i) - (df)(\mathcal{M}(e_i, e_i))\right).\end{aligned}$$

For the Equation (4.15) one first observes

$$\begin{aligned}(\nabla\nabla\omega)(A, B, X) = (DD\omega)(A, B, X) &- (D\omega)(\mathcal{M}(A, B), X) - (D\omega)(B, \mathcal{M}(A, X)) \\ &- \omega\left((D\mathcal{M})(A, B, X)\right) - (D\omega)(A, \mathcal{M}(B, X)) \\ &+ \omega\left(\mathcal{M}(\mathcal{M}(A, B), X)\right) + \omega\left(\mathcal{M}(B, \mathcal{M}(A, X))\right).\end{aligned}$$

Taking the negative trace in A and B then leads to

$$\begin{aligned}(\Delta^{\nabla}\omega)(X) = (\Delta^{g,D}\omega)(X) + (D\omega)(H, X) &+ \sum_i \epsilon_i (D\omega)(e_i, \mathcal{M}(e_i, X)) + \sum_i \epsilon_i \omega((D\mathcal{M})(e_i, e_i, X)) \\ + \sum_i \epsilon_i (D\omega)(e_i, \mathcal{M}(e_i, X)) &- \omega(\mathcal{M}(H, X)) - \sum_i \epsilon_i \omega\left(\mathcal{M}(e_i, \mathcal{M}(e_i, X))\right).\end{aligned}$$

The sum over $\omega((D\mathcal{M})(e_i, e_i, X)) = g((D\mathcal{M})(e_i, X, e_i), \omega^{\sharp})$ can again be replaced using Equation (4.8). The result is

$$\begin{aligned}(\Delta^{\nabla}\omega)(X) &= (\Delta^{g,D}\omega)(X) + (D\omega)(H, X) + \sum_i \epsilon_i (D\omega)(e_i, \mathcal{M}(e_i, X)) \\ &\quad - \mathrm{Ric}(X, \omega^{\sharp}) + \mathrm{Ric}^{g,D}(X, \omega^{\sharp}) + g(D_X H, \omega^{\sharp}) \\ &\quad + g(\mathcal{M}(X, H), \omega^{\sharp}) - \sum_i \epsilon_i g\left(\mathcal{M}(e_i, \mathcal{M}(X, e_i))\right) \\ &\quad + \sum_i \epsilon_i (D\omega)(e_i, \mathcal{M}(e_i, X)) - \omega(\mathcal{M}(H, X)) - \sum_i \epsilon_i \omega\left(\mathcal{M}(e_i, \mathcal{M}(e_i, X))\right) \\ &= (\Delta^{g,D}\omega)(X) - \mathrm{Ric}(X, \omega^{\sharp}) + \mathrm{Ric}^{g,D}(X, \omega^{\sharp}) + \omega(D_X H) \\ &\quad + (D\omega)(H, X) + 2\sum_i \epsilon_i (D\omega)(e_i, \mathcal{M}(e_i, X)) - 2\sum_i \epsilon_i \omega\left(\mathcal{M}(e_i, \mathcal{M}(e_i, X))\right).\end{aligned}$$

The third claim is a direct consequence of the Leibniz rule for connections acting on tensor products. ∎

Calculation of the difference of the Laplace operators in such a decomposition is motivated by the observation that the terms with coefficient of second-order derivative of g only appear in the Ricci tensor and in derivatives ∇H of the wave-gauge vector. A modification of the Laplace-Beltrami operator is given by the following definition.

Definition 4.1.11. Consider $S \in \mathcal{T}^{1,1}M$ to be a (1,1)-tensor field. Then a *modification* $\Delta + S$ of a Laplace operator Δ on tensor fields $T \in \mathcal{T}^{p,q}M$ is defined by

$$(\Delta^{\nabla} + S)T := \Delta^{\nabla} T + S \cdot T$$

with $S \cdot T$ being the Ricci product of S and T.

Observe that the so defined modification of a Laplacian will not change its behaviour if acting on smooth maps. Such a modification then is used for the definition of a reduced Laplacian.

Definition 4.1.12. Let $\Theta \in \mathcal{T}^{(1,1)}$ be is a tensor field on a Lorentzian manifold (M, g) not depending on derivatives of the metric g. A modification of the Laplace-Beltrami operator of the form

$$\left(\Delta^{\nabla} + \mathrm{Ric}^{\sharp} - DH + \Theta\right) : \mathcal{T}^{p,q}M \to \mathcal{T}^{p,q}M,$$

will be called *reduced Laplace operator with respect to D* or *reduced wave-operator*.

By Lemma 4.1.10 the principal part of a reduced Laplace operator is $\Delta^{g,D}$. Its coefficient is the metric tensor. Moreover the remaining coefficients do not depend on second-order derivatives of the metric. This is due to the fact that the only terms containing second-order derivatives have been the Ricci tensor and the derivative of the wave-gauge vector DH. Hence a PDE of type $\left(\Delta^{\nabla} + \mathrm{Ric}^{\sharp} - DH + \Theta\right) u = F[x, u, Du]$ can still be hyperbolic even if the metric is part of the set of unknowns u. On the other hand it will not guarantee hyperbolicity. An application is the method of conformal wave equations (section 4.3).

4.1.3. Einstein Equations

The problem of finding metrics that are Ricci-flat is to find solutions to the partial differential equations implied by $\mathrm{Ric} = 0$. It is formulated in terms of a torsion-free connection D acting on the metric by Equations (4.1) and (4.2). Unfortunately the system of equations is not hyperbolic. Then again the equations implied by the reduced Ricci tensor $\mathrm{Ric}^{(H)} = 0$ as defined above do allow a treatment as hyperbolic equations. In particular in Lorentzian signature they give a quasidiagonal, quasilinear systems of wave equations for the metric g [CB09]. In case where the initial data are given on a spacelike $n-1$ dimensional hypersurface, the Cauchy problem is widely studied (see for example [CB09, Ren08]). The Cauchy data are the induced metric on the hypersurface and and its first transversal derivatives, fulfilling the Einstein constraint equations, i.e. equations that necessarily have to be satisfied by Einstein metrics along the initial surface.

In addition the initial data are chosen such that the wave gauge vector H vanishes along the initial surface. For solutions of the reduced equation $\mathrm{Ric}^{(H)} = 0$ it then is shown by an energy inequality that H vanishes everywhere in the domain of dependence[1] of the initial hypersurface. As a consequence solutions to $\mathrm{Ric}^{(H)} = 0$ with vanishing wave-gauge vector H are metrics, which also fulfil the full Einstein equations $\mathrm{Ric} = 0$.

The main idea of this method is as follows. If the dimension is $n > 2$, then solutions to $\mathrm{Ric} = 0$ are also solutions to $G[g] = \mathrm{Ric} - \frac{\tau}{2} g = 0$ with the advantage of the second equation to be divergence-free, $\operatorname{div}\left(\mathrm{Ric} - \frac{\tau}{2} g\right) = 0$. Now the decomposition of the Ricci tensor into a reduced Ricci tensor and a rest with respect to some torsion-free connection D is entailed on to a corresponding decomposition of the scalar curvature

$$\tau = \tau^{(H)} + \mathcal{C}(DH), \tag{4.17}$$

where $\tau^{(H)} = \mathrm{tr}^{g} \mathrm{Ric}^{(H)}$ and $\mathcal{C}(DH)$ denotes the contraction of the $(1,1)$ tensor DH. This gives a decomposition of the Einstein tensor

$$G[g](X, Y) = G^{(H)}[g](X, Y) + \frac{1}{2}\left(g(X, D_Y H) + g(Y, D_X H) - \mathrm{tr}^{g} DHg(X, Y)\right). \tag{4.18}$$

1 One distinguishes between future and past domain of dependence. Let $S \subset M$ be a subset of the Lorentzian manifold (M, g). Then the future domain of dependence $D^{+}(S)$ is the set of all points p such that every inextendible past-directed curve starting in p intersects S. For the past domain of dependence, future-directed curves are considered.

The tensor $G^{(H)}(X,Y) = \mathrm{Ric}^{(H)} - \frac{1}{2}\,\mathrm{tr}^g\,\mathrm{Ric}^{(H)}$ analogous is called reduced Einstein tensor. Now assume that g is a solution to the reduced Einstein equation $\mathrm{Ric}^{(H)} = 0$ and let ∇ be the Levi-Civita connection associated to g. Then g also satisfies $G^{(H)} = 0$ and therefore implies vanishing divergence of $G[g] - G^{(H)}[g]$. In general $G[g]$ does not have to vanish but still is divergence-free. Hence the wave-gauge vector H satisfies at $p \in M$

$$\begin{aligned} 0 &= \operatorname{div}(G[g] - G^{(H)}[g])(X) \\ &= -\sum_i \epsilon_i \left(g\left(e_i, \nabla_{e_i} D_X H\right) + g\left(X, \nabla_{e_i} D_{e_i} H\right) - g\left(e_i, \nabla_X D_{e_i} H\right)\right) \end{aligned}$$

where $\{e_i\}$ is a local frame and X, e_i are local p-synchronous vector fields with respect to ∇. Now using

$$\nabla_X D_Y H - \nabla_Y D_X H = \mathrm{R}^D(X,Y)H + D_{[X,Y]}H + \mathcal{M}(X, D_Y H) - \mathcal{M}(Y, D_X H)$$

in p one gets

$$\begin{aligned} 0 = &\sum_i \epsilon_i g(X, D_{e_i} D_{e_i} H) + \sum_i \epsilon_i g\left(e_i, \mathrm{R}^g(e_i, X)H\right) \\ &+ \sum_i \epsilon_i \left(g\left(e_i, \mathcal{M}\left(e_i, D_X H\right) - \mathcal{M}\left(X, D_{e_i} H\right) + D_{[e_i,X]} H\right) - g\left(X, \mathcal{M}\left(e_i, D_{e_i} H\right)\right)\right). \end{aligned}$$

The last equation represents a linear hyperbolic PDE fulfilled by the wave-gauge vector H. Assuming g to be at least C^3, the second derivatives of H are well defined. Treatment of this type of equation is essential to the problem of solving the Einstein equations. Provided the wave-gauge vector H vanishes at the initial surface, it can be shown to vanish to the future domain of dependence of that surface. The case where the initial surface is spacelike is described for example in [FB52] and [CB09, Appendix III.3&III.4]. The case where the initial surface is a future null cone is for example treated in [CBMG10, Fri75]. The important method in both treatments is the existence of an energy inequality (see appendix C). Now let g be a solution to the reduced Einstein equation, with initial data that are prescribed on a spacelike hypersurface or on a future-directed null cone. If the initial data are chosen such that the wave gauge vector vanishes, the last considerations have some important consequence. At least locally the solution g will have vanishing wave gauge vector to the future of the initial surface. Hence it locally will be a solution to the full Einstein equations. The problem of prescribing the metric with vanishing wave gauge vector is treated in [CBCMG11b, CBCMG11a] in case of initial data on a characteristic cone.

4.1.4. Einstein Constraints

The terminus Einstein constraints have been and will be mentioned several times. Even if the explicit form will not be important to this thesis, we will give a short introduction here. The meaning of what Einstein constraints are strongly depends on the equations that are considered.

If Einstein or almost Einstein equations are treated as initial value problem, choosing adequate initial data are an essential part of the problem. In particular the initial data necessarily have to fulfil certain constraint equations to possibly be induced by an Einstein or almost Einstein metric. Usually the constraint equations can be distinguished by whether they are induced by the Einstein equations or almost Einstein equation on the initial surface or they are induced by the specific method of posing the initial value problem. The former are then called Einstein constraints. In the following, only Lorentzian metrics will be considered.

First consider the Einstein equation with maybe vanishing cosmological term $\mathrm{Ric} - \frac{\tau}{2}g + \Lambda g = 0$. Usually the data are assumed to be given on a Riemannian Cauchy surface Σ of the manifold under construction. Then the initial data on Σ consist of the induced Riemannian metric ς and the scalar valued second fundamental form $K(X,Y) = g\left(\mathrm{I\!I}(X,Y), e_0\right)$, where e_0 is the future-directed normal vector field on Σ. The latter corresponds to a transversal derivative of the induced metric[2]. The first constraint equations emerges from calculating $G[g]\,(e_0, e_0)$ along the initial surface. Using the Gauß equation for calculating the Ricci tensor along the initial surface gives

$$\tau^{\varsigma} - \|K\|^2_{\varsigma} + |\mathrm{tr}^{\varsigma} K|^2 - 2\Lambda = 0$$

for any tuple (ς, K) that is induced by an Einstein metric. These constraints are called Hamiltonian constraints. A second equation is gained by calculating the tensor induced by $G[g]\,(\cdot, e_0)$ on the initial surface Σ. The Codazzi equations then gives

$$\mathrm{div}^{\varsigma} K - d\,(\mathrm{tr}^{\varsigma} K) = 0,$$

which are called the momentum constraints.

Now consider the problem where solutions g to the vacuum Einstein equation with vanishing cosmological term are wanted. If the initial data are given on intersecting characteristic hypersurfaces or on a characteristic cone, the induced metric on the hypersurface degenerates. For the corresponding system of PDEs, the initial data are given in a neighbourhood of the intersection of the two hypersurfaces or in a neighbourhood of the cones vertex. The initial data are the full metric and has to fulfil certain constraint equations. The explicit form of the constraints depends on the splitting that is used to decompose the Einstein equations to a set of evolution equations and constraint equations along the characteristic hypersurfaces. By choice of a null vector field N_1 on the cone or two null vector fields N_1, N_2 on the intersecting characteristic hypersurfaces the constraints for the vacuum field equations emerge by requiring

$$G\,[g]\,(N_i, N_i) = 0 \qquad\qquad G\,[g]\,(N_i, \cdot) = 0$$

along the initial hypersurfaces. An additional set of constraint equations results from the fact that only the reduced Einstein equations are solved and one has to assure that the initial data provide solutions that also solve the full Einstein equations. Depending on the problem, it may be necessary to specify an additional function as initial data. For a more comprehensive picture of the characteristic initial value problem for Einstein equations we refer for example to [CP12].

If almost Einstein metrics with positive almost scalar curvature are evaluated from an initial value problem with initial data on the Riemannian singularity set, the constraints can be given in terms of coefficients of the Fefferman-Graham expansion [And04, dHSS01]. The manifold locally is assumed to be of the form $\left(I \times \Sigma, g = -dt^2 + g_t\right)$ for a geodesic parameter t and a family of metrics g_t on Σ. In this splitting one has $[\partial_t, X] = [\partial_t, \nabla_{\partial_t} X] = 0 = \nabla_{\partial_t}\partial_t$ for vector fields X tangent to the slices of constant parameter t. Calculation of $g\,(\mathrm{R}(X, \partial_t)\partial_t, Y)$ yields a Riccati equation, namely $g\,(\mathrm{R}(X, \partial_t)\partial_t, Y) = -\left(\mathcal{L}_{\partial_t} K\right)(X, Y) + \left(\mathrm{tr}^{g_t}_{23} K \otimes K\right)(X, Y)$ for the scalar valued second fundamental form $K(X, Y)$. The Codazzi equation on the other hand yields $dH + \mathrm{div}^{g_t} K = 0$. Now, if g_t is expanded at $t = 0$ in powers of t, the latter equations provide constraint equations for the initial data on the hypersurface $\{t = 0\}$. In particular by requiring $\mathrm{Ric}[t^{-2}g] + t^{-2}g = 0$, the expansion is determined by prescribing the induced metric $\gamma = g_{(0)}$ at the initial surface and the $(n-1)$-th coefficient $g_{(n-1)}$ [And04]. The possible choice of $\left(\gamma, g_{(n)}\right)$ is restricted by

$$\mathrm{tr}^{\gamma}\, g_{(n-1)} = \omega_1 \qquad\qquad \mathrm{div}^{\gamma}\, g_{(n-1)} = \omega_2,$$

where ω_1 and ω_2 are fully determined by γ and its derivatives. In case that n is even one has $\omega_1 = 0$ and $\omega_2 = 0$.

For almost Einstein metrics with vanishing almost scalar curvature and initial data in a neighbourhood of a characteristic cone at conformal infinity a set of constraint equations is calculated for a associated system of PDEs in [Pae13].

2 The correspondence of the second fundamental form to transversal derivatives of the induced metric becomes apparent, if one considers the thickening $I \times \Sigma$ with metric $g = -dt^2 + \varsigma_t$, where ς_t is a family of Riemannian metrics on Σ such that $\varsigma_0 = \varsigma$. Then using the Koszul formula, it holds $g\,(\nabla_X Y, \partial_t) = -\frac{1}{2}\left(\mathcal{L}_{\partial_t}\varsigma_t\right)(X, Y)$ for vector fields that are tangent to the hypersurfaces $\{t\} \times \Sigma$ and hence $K = -\frac{1}{2}\mathcal{L}_{\partial_t}\varsigma_t =: \dot{\varsigma}_t$.

4.2 FRIEDRICH'S CONFORMAL FIELD EQUATIONS

If solutions (g, σ) to the almost Einstein equation $A[g, \sigma] = 0$ are to be found, one has to deal with the singular behaviour of that equation at points where σ vanishes. The corresponding PDE has a principal term that vanishes where σ does. Hence the system of equations degenerates at the set $\Sigma = \sigma^{-1}(0)$. In [Fri81b] H. Friedrich introduced a method for reducing the degenerate system of equations to a first-order, quasilinear, symmetric hyperbolic system of partial differential equations. The method described there is for 4 dimensions but also applies to higher dimensions. Unfortunately the system is no longer symmetrisable in higher dimension. The main idea is to treat g, σ and in addition some of their derivatives as independent variables to the system. This gives a reduced first-order system of differential equation that does not degenerate, even where σ vanishes.

Consider (M, D) to be a smooth manifold with torsion-free connection D. One now considers the set of unknowns $u := (\sigma, \varsigma, \rho, g, \mathrm{P}, w, \mathcal{M})$, where σ and ρ are real valued maps, ς is a one form on M, g and P are symmetric (2,0)-tensors, w is a (3,1)-tensor with algebraic properties of the Weyl tensor[3] and $\mathcal{M}$ is a $(2, 1)$-tensor symmetric in its contravariant arguments. Hence $\nabla := D + \mathcal{M}$ again is a torsion-free connection. Let X, Y, Z be vector fields on M and $\{e_i\}$ be a local orthonormal frame with respect to some arbitrary metric h. The linear system of PDEs for the variable u found by H. Friedrich locally is given by

$$
\begin{array}{lrcl}
(i) & D\sigma & = & \varsigma \\
(ii) & D\varsigma & = & -\varsigma\left(\mathcal{M}(\cdot,\cdot)\right) - \sigma\mathrm{P} - \rho g \\
(iii) & D\rho & = & \varsigma \lrcorner \mathrm{P} \\
(iv) & (D\mathcal{M})(Y,X,Z) - (D\mathcal{M})(X,Y,Z) & = & \mathrm{R}^D(X,Y)Z + \mathcal{M}(\mathcal{M}(X,Z),Y) - \mathcal{M}(\mathcal{M}(Y,Z),X) \\
 & & & -\sigma^{n-3}w(X,Y)Z - [(\mathrm{P} \owedge g)(X,Y,Z,\cdot)]^{\sharp g} \\
(v) & (Dg)(X,Y,Z) & = & g(\mathcal{M}(X,Y),Z) + g(\mathcal{M}(X,Z),Y) \\
(vi) & (D\mathrm{P})\,(X,Y,Z) - (D\mathrm{P})\,(Y,X,Z) & = & \mathrm{P}\,(X,\mathcal{M}(Y,Z)) - \mathrm{P}\,(Y,\mathcal{M}(X,Z)) \\
 & & & -\sigma^{n-4}\varsigma\,(w(X,Y)Z) \\
(vii) & (\mathcal{C}Dw)(X,Y,Z) & = & -\sum_i \epsilon_i h\,(e_i, \mathcal{M}(e_i, w(X,Y)Z)) \\
 & & & +\sum_i \epsilon_i h\,(e_i, \mathcal{M}\,(Z, w(X,Y)e_i)) \\
 & & & +\sum_i \epsilon_i h\,(e_i, \mathcal{M}\,(X, w(e_i,Y)Z)) \\
 & & & +\sum_i \epsilon_i h\,(e_i, \mathcal{M}\,(Y, w(X,e_i)Z))
\end{array}
$$

where $\kappa \in \mathbb{R}$ is a constant arbitrary term and $(\mathcal{C}Dw)(X, Y, Z)$ is the contraction, which can be given in terms of the metric h by

$$(\mathcal{C}Dw)(X,Y,Z) := \sum_i \epsilon_i h\,(e_i, (Dw)(e_i, X, Y)Z)\,.$$

The contractions of the $(5, 2)$-tensor $\mathcal{M} \otimes w$ on the right-hand side are given in terms of a orthonormal frame $\{e_i\}$ with respect to h. Nevertheless Equation (vii) does not depend on the metric h.

A solution $u = (\sigma, \varsigma, \rho, g, \mathrm{P}, w, \mathcal{M})$ of the *conformal field equations*[4] (i)-(vii) gives rise to a asymptotically flat solution of the almost Einstein equation $A[g, \sigma]$ by [Fri83, Theorem 3.1] and [Fri82]. One then has the following statement in a notation that is compatible with the notation in this thesis.

Theorem *Suppose u is a solution of the conformal field equations (i)-(vii), then, if $0 = 2\sigma\rho + \|\varsigma\|_g^2$ holds at one point of M, it is satisfied everywhere on M. Furthermore (M, g, σ) is an almost Einstein structure with vanishing almost scalar curvature.*

As pointed out before, the theorem is a consequence of the underlying almost Einstein equations. Since the potential $\mathcal{M}$ is supposed to be symmetric in its covariant arguments, the new

3 The algebraic properties are vanishing of all possible contractions and $w(X, Y) = -w(Y, X)$. At the initial surface one also requires $g\,(w(X, Y)V, W) = g\,(w(V, W), X, Y)$.

4 The term regular or reduced conformal field equations is also used in literature.

connection $\nabla := D + \mathcal{M}$ will be torsion-free. Equation (v) then guarantees that in addition it is compatible with the metric. By uniqueness ∇ must coincide with the Levi-Civita connection of g. The remaining equations can now be rewritten to

$$\begin{array}{lrcl}
(i) & \nabla\sigma & = & \varsigma \\
(ii) & \nabla\varsigma & = & -\sigma\mathrm{P} - \rho g \\
(iii) & \nabla\rho & = & \varsigma \lrcorner \mathrm{P} \\
(iv) & \mathrm{R}^{\nabla}(X,Y)Z & = & \sigma^{n-3} w(X,Y)Z + (\mathrm{P} \owedge g)(X,Y,Z,\cdot)^{\sharp g} \\
(vi) & (\nabla\mathrm{P})(X,Y,Z) - (\nabla\mathrm{P})(Y,X,Z) & = & -\sigma^{n-4}\varsigma\,(w(X,Y)Z) \\
(vii) & (\mathrm{div}^g\, w)(Z,X,Y) & = & 0,
\end{array}$$

where R^{∇} is the Riemann curvature tensor with respect to g. By taking the g-trace of (vi), using Equation (1.36) for the trace of the Kulkarni-Nomizu product and considering the algebraic properties of w one finds $\mathrm{Ric}^g = \left(\mathrm{tr}_g \mathrm{P}\right) g + (n-2)\mathrm{P}$. Consequently P is the Schouten tensor of g. Reconsidering Equation (vi) with that information and using uniqueness of the decomposition of the Riemann tensor, $\mathrm{W} := \sigma^{n-3} w$ has to be the $(3,1)$-Weyl curvature of g. Consequently W will be zero, where σ is.

Corollary *Let (g,σ) be part of a solution to the conformal field equations, then $(M \setminus \Sigma, \sigma^{-2} g)$ asymptotically is a conformally flat manifold, i.e. has asymptotically vanishing Weyl curvature as σ tends to zero.*

Using Equation (i), the second equation can be rewritten to $0 = \nabla d\sigma + \sigma\mathrm{P} + \rho g$. Taking the g-trace of the last equation yields $\rho = \frac{1}{n}(\Delta^g\sigma - J\sigma)$. The result then is $A[g,\sigma] = 0$ and consequently (M, g, σ) is an almost Einstein structure. Consequently if $0 = 2\sigma\rho + \|\varsigma\|_g^2 = S[g,\sigma]$ holds at one point of the manifold, it holds everywhere by Corollary 1.4.11.

The remaining equations are redundant but necessary to have a first-order system. Already knowing that (M, g, σ) is an almost Einstein structure, Equation (vii) reflects the property of $\sigma^{-2} g$ being Einstein away from Σ and it coincides with Equation (1.111) ($\mathrm{div}\,\sigma^{3-n}\,\mathrm{W} = 0$) if written in terms of the Weyl tensor. Equation (vi) reflects the same property, it is an application of Equation (1.125) where the Cotton tensor is written in terms of the Schouten tensor. Equation (iii) is a consequence of Lemma 1.4.9.

Remark. On the other hand, if (g,σ) is a solution to the almost Einstein equation $A[g,\sigma] = 0$, then this gives rise to a solution of the conformal field equations only if the rescaled Weyl tensor $\sigma^{3-n}\,\mathrm{W}$ extends to $\Sigma = \sigma^{-1}(0)$ at least of class C^1. In that case, let be D a torsion-free connection on on M. Then $u = \left(\sigma, d\sigma, \frac{1}{n}\left(\Delta^g\sigma - \frac{\tau^g}{2(n-1)}\sigma\right), g, \mathrm{P}^g, \sigma^{3-n}\,\mathrm{W}, \nabla^g - D\right)$ is a solution to the *conformal field equations* (i)-(vii).

It was shown by H. Friedrich that in dimension $n = 4$ the PDEs provide a symmetrisable hyperbolic first-order system [Fri81a, Fri81b, Fri82], i.e. decomposes into a system of hyperbolic symmetric system of evolution equations and constraint equations[5]. Also different types of initial data have been discussed. The restriction to four dimensions is essential to the hyperbolicity of the equation. It is based on the decomposition behaviour of the Weyl tensor. The Weyl tensor fulfils a *Bianchi type equation* [Ale12, Equation 2.68]:

$$\mathcal{B}(\nabla\,\mathrm{W}) = \sum \mathcal{C}(\mathrm{div}\,\mathrm{W} \otimes g), \tag{4.19}$$

where $\mathcal{B}$ is the *Bianchi operator* on (p,q)-tensors with $p \geq 3$, defined by $(\mathcal{B}T)(X,Y,Z,\cdot,\dots) := \frac{1}{3}\left(T(X,Y,Z,\cdot,\dots) + T(Z,X,Y,\cdot,\dots) + T(Y,Z,X,\cdot,\dots)\right)$. The term on the right-hand side represents a linear combination of permutations of $\mathrm{div}\,\mathrm{W} \otimes g$. Provided $\tilde{g}$ is an Einstein metric, the left-hand side vanishes. Taking the divergence of $\mathcal{B}\tilde{\nabla}\tilde{\mathrm{W}}$ then gives [Ale12]

$$\Delta^{\tilde{g}}\tilde{\mathrm{W}} = Q(\tilde{\mathrm{W}}) + L(\tilde{\mathrm{W}}),$$

5 A system $B^0\partial_0 u + \sum_{j=1,..m} B^j\partial_j u = f$ is called symmetric, provided the matrices B^j are symmetric (see for example [Eva98, Chapter 7.3.2])

where $Q(\tilde{W})$ is a term quadric in $\tilde{W}$ and $L(\tilde{W})$ a term linear in $\tilde{W}$[6]. In Lorentzian signature the above system provides a hyperbolic system of equations, which is satisfied by the Weyl tensor $\tilde{W}$. A special property in 4 dimensions is the equivalence of the Bianchi equation $\mathcal{B}(\nabla \mathrm{W}) = 0$ and the contracted Bianchi equation $\operatorname{div} \mathrm{W} = 0$[7]. The latter is equivalent to $\operatorname{div} w = 0$ by Equation (1.111) and that in principle is why imposing $\operatorname{div} w = 0$ in the conformal field equations suffices to get an hyperbolic system. The advantage of using $\operatorname{div} w = 0$ in place of $\mathcal{B}(\tilde{\nabla}\tilde{\mathrm{W}}) = 0$ is that it is regular where σ vanishes. Using a spin frame formalism, H. Friedrich was able to split the conformal field equations into a system of symmetric hyperbolic evolution equations and a set of constraint equations [Fri81a, Fri81b, Fri82, Fri83]. He pointed out that in higher dimension usage of the full Bianchi equations or a conformal analogue may be necessary to get hyperbolic regular PDEs [Fri02]. An ansatz for using the full Bianchi equations will be presented in the next section.

4.3 CONFORMAL WAVE EQUATIONS

Related to the conformal field equations there have been several results concerning uniqueness and existence issues. Some have been mentioned in the introduction of this thesis. A particular interest is the Cauchy problem with initial data given on a characteristic cone that also acts as conformal infinity. A recent result is the construction of a system of quasilinear wave equations, which corresponds to the conformal field equations [Pae13]. Existence of solutions can be proven if the characteristic initial data at conformal infinity satisfy some smoothness conditions [CP13]. This section will summarise the construction and will provide an ansatz to generalise it to higher even dimensions.

The conformal field equations written in the form

$$\begin{array}{llcl}
(ii) & \nabla d\sigma & = & -\sigma \mathrm{P} - \rho g \\
(iii) & \nabla \rho & = & \operatorname{grad}^g \sigma \lrcorner \mathrm{P} \\
(iv) & \mathrm{R}^{\nabla}(X,Y)Z & = & \sigma^{n-3} w(X,Y)Z + (\mathrm{P} \owedge g)(X,Y,Z,\cdot)^{\sharp g} \\
(vi) & (\nabla \mathrm{P})(X,Y,Z) - (\nabla \mathrm{P})(Y,X,Z) & = & -\sigma^{n-4} d\sigma\,(w(X,Y)Z) \\
(vii) & (\operatorname{div}^g w)(Z,X,Y) & = & 0.
\end{array}$$

are the starting point of the construction. The first of the conformal field equations has been used to remove the unknown ς in the remaining equations. A second ingredient of the construction is usage of a generalised wave-map gauge (compare section 4.1). Considers two metrics g and $\hat{g}$ on M with Levi-Civita connections $\nabla = \nabla^g$ and $D = \nabla^{\hat{g}}$. The *generalised wave-gauge vector* then is defined as

$$H = \operatorname{tr}^g \mathcal{M} - V,$$

where $\mathcal{M} = \nabla - D$ is the potential and V is a vector field that may explicitly depend on $x \in M$ and the unknowns u. The explicit set of unknowns that is denoted by u depends on the system under consideration. The reduced Ricci tensor in a generalised wave-map gauge is defined as

$$\operatorname{Ric}^{(H)}(X,Y) := \operatorname{Ric}(X,Y) - \frac{1}{2}\left(g(X, D_Y H) + g(Y, D_X H)\right). \qquad (4.20)$$

6 an explicit formula is given for example in [Fri02]

7 To see this property one considers the $(2,2)$-Weyl tensor as map $M\ \mathrm{W} : \Gamma(\Lambda^2 T^*M) \to \Gamma(\Lambda^2 T^*M)$ on antisymmetric tensor fields. In four dimensions the Weyl tensor decomposes into a self-dual part W^+ and an anti-self-dual part W^-, with $* \mathrm{W}^\pm = \pm \mathrm{W}^\pm$. Now let $d^\nabla : \Gamma\left(\Lambda^p T^*M \otimes \Lambda^2 T^*M\right) \to \Gamma\left(\Lambda^{p+1} T^*M \otimes \Lambda^2 T^*M\right)$ be the exterior differential associated to the Levi-Civita connection of g [Bes08, (1.12)]. Then locally one has

$$\mathcal{B}(\nabla \mathrm{W}) = d^\nabla \mathrm{W} = - * \circ \delta^\nabla \circ * \mathrm{W} = - * \left(\delta^\nabla \mathrm{W}^+ - \delta^\nabla \mathrm{W}^-\right),$$

where δ^∇ is the divergence. Linear independence of $\delta^\nabla \mathrm{W}^+$ and $\delta^\nabla \mathrm{W}^+$ bijectivity of the Hodge dual then gives the equivalence.

As mentioned before V will be allowed to depend on g but not on its higher derivatives. Hence the second term still removes second-order derivatives of g, which would prevent the principal part of Ric from being a wave-like equation on g. The operator

$$\square^{(H)} := \Delta^{\nabla} + \mathrm{Ric}^{\sharp} - DH - (n-2)\mathrm{P}^{\sharp} - J\,\mathrm{id} \tag{4.21}$$

so still is a reduced wave operator (compare definition 4.1.12 for the notation) and may entail a hyperbolic equation even if g is part of the unknowns. J is considered to be an arbitrary map, which locally can be prescribed to the system and will later coincide with the trace of P. Important to the following consideration is the observation that the modification of the Laplacian on the right-hand side can be rewritten to

$$\begin{aligned}\mathrm{Ric}(X,Y) - g(D_X H, Y) - (n-2)\mathrm{P}(X,Y) - Jg(X,Y) &= \\ \mathrm{Ric}^{(H)}(X,Y) + \frac{1}{2}\left(g(X, D_Y H) - g(Y, D_X H)\right) &- (n-2)\mathrm{P}(X,Y) - Jg(X,Y).\end{aligned} \tag{4.22}$$

It is important due to the following observation. Let g and P be solutions to an arbitrary set of PDEs and assume that for that solution g is a Lorentzian metric and P its Schouten tensor. Assume further g to have vanishing wave gauge vector with respect to D. Then $\square^{(H)}$ coincides with Δ^{∇} and solutions to equations involving $\square^{(H)}$ will also be solutions to the same equations, where the reduced operator is replaced by the Laplacian Δ^{∇}. Let for example T be a tensor field that is a solution to an equation of type $\square^{(H)}T = \dots$, where $\square^{(H)}$ is an operator in D. Then the previous assumptions assure that T also is a solution to Δ^{∇}. Moreover in place of considering a equation that correspond to $\mathrm{Ric}[g]$ is suffice to impose an equation on $\mathrm{Ric}^{(H)}[g]$ and to assure that a solution g has vanishing wave-gauge vector. This is quite important, since equations involving $\mathrm{Ric}^{(H)}$ are better behaved in terms of D-derivatives of g.

Consider the unknown $u = (g, \mathrm{P}, w, \sigma, \rho)$. By differentiating the conformal field equations, a system of Laplace-type equations can be derived [Pae13]

$$\begin{aligned}\Delta^{\nabla}(\mathrm{P}, w, \sigma, \rho) &= F(x, u, \nabla u) \\ \mathrm{Ric} &= (n-2)\mathrm{P} + Jg.\end{aligned} \tag{4.23}$$

This system then is replaced by the reduced system

$$\begin{aligned}\square^{(H)}(\mathrm{P}, w, \sigma, \rho) &= F(x, u, \nabla u) \\ \mathrm{Ric}^{(H)}[g] &= (n-2)\mathrm{P} + Jg.\end{aligned} \tag{4.24}$$

The latter system is called *conformal wave equations*. The name is a reference to the fact that the reduced Ricci $\mathrm{Ric}^{(H)}[g]$ and $\square^{(H)}$ have leading order term $\mathrm{tr}^g\, DDg = \Delta^{g,D}$. Solutions to the conformal field equations with vanishing wave-gauge vector H will by constructions also be solutions to the conformal wave equations. A first result then is that provided a set of conditions is fulfilled by the initial data, then solutions to the conformal wave equations, implying the initial data on the initial characteristic set, will also be solutions to the conformal field equations [Pae13, Theorem 3.7]. Assuming the initial set is a null cone with respect to the metric g to be constructed and the zero-set of σ, the second goal is to provide a set of constraint equations to the initial data in a certain wave-map gauge, which guarantees that solutions to the conformal wave equations are also solutions to the conformal field equations and vice versa ([Pae13, Theorem 5.1]). An important tool to derive the constraint equations is the usage of *adapted null coordinates* that provide a way to derive a hierarchical system of algebraic and ordinary differential equations along null rays originating from the vertex of the cone.

The conformal field equations eliminate the degeneracy of the almost Einstein equations by imposing new unknowns and new equations to the system. In particular the unknown w of a solution is the rescaled Weyl tensor. Consequently solutions to the conformal field equations will have vanishing Weyl tensor on the conformal boundary and moreover its asymptotic behaviour has to be such that $\sigma^{3-n}\,\mathrm{W}$ has a sufficiently smooth extension to the boundary. This requirements are quite strong restrictions to the set of possible solutions to the almost Einstein equations.

A second type of wave equations can be derived, where the requirement of vanishing Weyl curvature is weakened. To keep the equivalence between solutions to the wave equations and solutions to the conformal field equations, one needs stronger restrictions to the initial data. The ansatz is to replace the unknowns $(g, \mathrm{P}, w, \sigma, \rho)$ by a new set of unknowns $(g, \mathrm{P}, \mathrm{C}, \mathrm{W}, \sigma, \rho)$ and to impose a set of Laplace-type equations on them. W is considered to have the algebraic properties of the Weyl tensor and C is considered to have the algebraic properties of the Cotton tensor. Construction of the system of wave-equations is sketched in the following.

Assume g is a metric, ∇ its Levi-Civita connection, P the Schouten tensor, C the Cotton tensor and W the Weyl tensor. Then by Equation (1.52), $(g, \mathrm{P}, \mathrm{C}, \mathrm{W})$ satisfies the equation

$$\Delta^{\nabla}\mathrm{W} = -(\Delta^{\nabla}\mathrm{P} + \mathrm{Hess}^{g} J) \owedge g + F[\mathrm{W}, \mathrm{C}, \nabla\mathrm{C}, \mathrm{P}, g], \tag{4.25}$$

where $F[\mathrm{W}, \mathrm{C}, \nabla\mathrm{C}, \mathrm{P}]$ is a rational algebraic term depending only on W, C, ∇C and P. For the Cotton tensor Equation (1.53) provides a similar condition

$$\begin{aligned}(\Delta^{\nabla}\mathrm{C})(X, Y, Z) = &\left(\nabla\left(\Delta^{\nabla}\mathrm{P} + \mathrm{Hess}^{g} J\right)\right)(Y, Z, X)\\ &- \left(\nabla\left(\Delta^{\nabla}\mathrm{P} + \mathrm{Hess}^{g} J\right)\right)(Z, Y, X) + F[\mathrm{W}, \mathrm{C}, \mathrm{P}, \nabla\mathrm{P}, g].\end{aligned} \tag{4.26}$$

The dependence on J and its derivatives is partially suppressed on the right-hand side. An equation involving the Laplace of the Schouten tensor then in even dimension n is provided by the obstruction tensor as for example given in [GH05][8]

$$\mathcal{O}[g] = \left(\Delta^{\nabla}\right)^{\frac{n}{2}-2}\left(\Delta^{\nabla}\mathrm{P} + \mathrm{Hess}^{g} J\right) + F^{n-1}, \tag{4.27}$$

The term F refers to derivatives of the metric of order less or equal to $n-1$. Vanishing of the obstruction tensor gives an equation that necessarily has to be fulfilled to get an almost Einstein manifold.

Now the remaining Laplace-type equations for σ, ρ and the metric g are provided by the demand that (M, g, σ) has to be an almost Einstein structure. The definition gives $n\rho = \Delta\sigma - J\sigma$ and hence an equation to σ. Equation (1.122) provides an equation to ρ and $\mathrm{Ric}[g] = (n-2)\mathrm{P} + Jg$ corresponds to an equation on g. There are two problems remaining, if one is interested in hyperbolic equations on the unknowns $(g, \mathrm{P}, \mathrm{C}, \mathrm{W}, \sigma, \rho)$. The first is that the equations on the Cotton and Weyl tensors do involve second- and third-order derivatives of the Schouten tensor on the right-hand side and the Obstruction tensor gives only a Laplace-type equation on the Schouten tensor in 4 dimensions. The second challenge is that if g is considered to be part of the unknowns, even the Laplace operator with respect to ∇ on the left-hand side is not diagonal in its leading order term. The second problem can be solved by replacing the Laplace operator by the reduced d'Alembertian introduced in Equation (4.21) and by using the reduced Ricci given in Equation (4.20). The emerging Laplacian $\Delta^{g,D}$ with respect to some connection D then is diagonal in its leading order term. Part of this method is to impose vanishing of the wave-gauge vector H on the initial data. It then has to be shown that vanishing of H is propagated to full solution, i.e. that solutions to the reduced system with such initial data automatically have vanishing wave-gauge vector.

The first problem only appears in even dimension $n > 4$. In 4 dimensions, the Bach tensor is the obstruction tensor and for example given in Equation (1.45). Its vanishing implies that

$$\Delta^{\nabla}\mathrm{P} + \mathrm{Hess}^{g} J = F[\mathrm{P}, g]$$

holds for almost Einstein structures, where $F[\mathrm{P}, g]$ only depends on the Schouten tensor and the metric. At first sight, this gives an equation to the Schouten tensor. To get rid of the Hessian of

8 If compared to the first appearance of the obstruction tensor in Equation (3.5), this time the dimension of M is n. This explains the different exponents.

J, one uses that at least locally J may be prescribed to an almost Einstein structure by exploiting conformal covariance. So by fixing J, $\mathrm{Hess}^g J$ just becomes an inhomogenity on the right-hand side, which will coincide with the trace of P for solutions to the final system. The Laplacian on the left-hand side then is replaced by the reduced d'Alembertian. Moreover this equation provides a method to substitute the second- and third-order derivatives on the right-hand side of the equations for W and C by at most first-order derivatives of g and P. As a matter of fact, the new derivatives of P can be replaced by C due to the antisymmetrisation that is involved [Pae13]. Finally in $n = 4$ dimensions this gives a reduced system

$$\begin{aligned} \Box^{(H)}(\mathrm{P}, \mathrm{C}, \mathrm{W}, \sigma, \rho) &= F[x, u, Du] \\ \mathrm{Ric}^{(H)} &= (n-2)\mathrm{P} + Jg, \end{aligned} \tag{4.28}$$

where the set of unknowns is $u = (g, \mathrm{P}, \mathrm{C}, \mathrm{W}, \sigma, \rho)$. The system is referred to as *alternative system of conformal wave equations*. [Pae13, Theorem 6.4] then states that provided the initial data fulfil a set of constraint equations on the initial surface[9], then solutions to the reduced system will also solve the system

$$\begin{aligned} \Delta^{\nabla}(\mathrm{P}, \mathrm{C}, \mathrm{W}, \sigma, \rho) &= F[x, u, \nabla u] \\ \mathrm{Ric}[g] &= (n-2)\mathrm{P} + Jg. \end{aligned} \tag{4.29}$$

Moreover it is shown that this theorem also applies to initial data on a characteristic cone, which by construction is the zero set of σ.

An interesting question is to what extent this method can be generalised to arbitrary dimension. A method may be implied in even dimensions, where in 4 dimensions the Bach tensor is used to get rid of higher-order derivatives of P. In higher even dimension the obstruction tensor has to be used to get a set of equations of Laplace type. This evidently involves powers of the Laplace operator. In particular vanishing of the obstruction tensor gives an equation for $\Delta^{\frac{n}{2}-1}\mathrm{P}$ with $\Delta = \Delta^{\nabla}$ for the moment. Let $n \geq 6$ be even. To get rid of the terms involving $(\Delta\mathrm{P} + \mathrm{Hess}\, J)$ on the right-hand side of the equations for Cotton and Weyl tensors, one will have to consider higher powers of the Laplacian. In case of Equation (4.25) this reads as

$$\Delta^{\frac{n}{2}-1}\mathrm{W} = -\left(\Delta^{\frac{n}{2}-2}(\Delta\mathrm{P} + \mathrm{Hess}^g J)\right) \owedge g + F[\nabla^{n-4}\mathrm{W}, \nabla^{n-3}\mathrm{C}, \nabla^{n-4}\mathrm{P}, g], \tag{4.30}$$

where ∇^k just denotes the highest derivative of the specific tensor. The situation is little more complicated in case of the equation for the Cotton tensor, since one has to commute the operator with the covariant derivative on the right-hand side. The initial situation is the equation

$$\begin{aligned} \left(\Delta^{\frac{n}{2}-1}\mathrm{C}\right)(X, Y, Z) = &\left(\Delta^{\frac{n}{2}-2}\left(\nabla\left(\Delta\mathrm{P} + \mathrm{Hess}\, J\right)\right)\right)(Y, Z, X) \\ &- \left(\Delta^{\frac{n}{2}-2}\left(\nabla\left(\Delta\mathrm{P} + \mathrm{Hess}\, J\right)\right)\right)(Z, Y, X) \\ &+ F[\nabla^{n-4}\mathrm{W}, \nabla^{n-4}\mathrm{C}, \nabla^{n-3}\mathrm{P}, g]. \end{aligned} \tag{4.31}$$

Using (1.55) and Lemma 1.1.14 for commuting the Laplacian with the covariant derivative on tensors, the remaining terms will involve only derivatives of P, C and W of highest order $n-4$ and hence one is left with an equation of type

$$\begin{aligned} \left(\Delta^{\frac{n}{2}-1}\mathrm{C}\right)(X, Y, Z) = &\left(\nabla\left(\Delta^{\frac{n}{2}-2}\left(\Delta^{\nabla}\mathrm{P} + \mathrm{Hess}\, J\right)\right)\right)(Y, Z, X) \\ &- \left(\nabla\left(\Delta^{\frac{n}{2}-2}\left(\Delta^{\nabla}\mathrm{P} + \mathrm{Hess}\, J\right)\right)\right)(Z, Y, X) \\ &+ F[\nabla^{n-4}\mathrm{W}, \nabla^{n-4}\mathrm{C}, \nabla^{n-3}\mathrm{P}, g]. \end{aligned} \tag{4.32}$$

Now terms involving $\Delta^{\frac{n}{2}-1}\mathrm{P}$ on the right-hand side of Equations (4.30) and (4.32) can be eliminated by demanding a vanishing obstruction tensor on almost Einstein manifolds. Vanishing of the obstruction tensor itself gives an equation to $\Delta^{\frac{n}{2}-1}\mathrm{P}$ with lower order derivatives of Schouten, Cotton and Weyl tensors on the right-hand side.

9 For example the wave gauge vector, its covariant derivative and $2\sigma\rho + \|d\sigma\|_g^2 (= S[g, \sigma])$ have to vanish.

The conjecture is that by introducing powers $\Delta^k P$, $\Delta^k C$ and $\Delta^k W$ as new variables to the system, it can be rewritten as a system of wave-like equations as it has been done in case of a system for the metric and the Anderson-Fefferman-Graham equations in [AC05]. A further analysis of this conjecture is beyond the immediate intention of this thesis and left to a future investigation.

5 ALMOST EINSTEIN STRUCTURES WITH VANISHING ALMOST SCALAR CURVATURE

The main focus of this thesis is on almost Einstein structures in Lorentzian signature $(-+\cdots+)$ with vanishing almost scalar curvature. Throughout this chapter (M, g, σ) will be such a structure with $S[g,\sigma] = 0$. If not specified differently, $\nabla = \nabla^g$ will be the Levi-Civita connection with respect to g. The chapter will start with a short analysis of the differentiability of the defining function σ. Next, the structure of its zero set Σ is considered. The critical points of σ along the zero set will be identified as focal points and vanishing points of the Weyl tensor. The analysis of Σ is completed by a section about the asymptotic behaviour of $\sigma^{-2}g$-geodesics near the conformal boundary Σ. This implies an approach to regain similar results on conformally compactified spacetimes in the context of almost Einstein structures. Finally, a set of special coordinates is constructed, which is adapted to the local topology of the singularity set Σ. The coordinates represent a compromise between Morse coordinates, in which the map σ has a particularly simple form, and geodesic coordinates, in which radial geodesics have a most simple form. They in particular combine the properties of keeping the simple form of σ in Morse coordinates and at the same time ensure a simple form of radial null geodesics. This compromise on the other hand is accompanied by a loss of differentiability at the origin and the loss of a generic affine parametrisation of radial null geodesics. The simple characterisation of radial timelike and spacelike geodesics is also lost. Finally, the form of an almost Einstein metric will be calculated in such coordinates.

5.1 TOPOLOGY OF THE CONFORMAL BOUNDARY Σ

5.1.1. Basic Properties of Σ

Let (M, g, σ) be an almost Einstein structure. Then from vanishing of the almost scalar curvature $S[g,\sigma]$ one gets $\|\operatorname{grad}\sigma\|^2 = -2\sigma\rho$. Hence the gradient $\operatorname{grad}\sigma$ is a null vector for points at the singularity set Σ. We decompose Σ into two disjoint sets

$$\begin{aligned} \Sigma_c &:= \{p \in \Sigma \mid d\sigma_p \neq 0\} \\ \Sigma_d &:= \{p \in \Sigma \mid d\sigma_p = 0\}, \end{aligned}$$

such that $\Sigma = \Sigma_c \dot{\cup} \Sigma_d$. The set Σ_d will turn out to be a set of isolated points, in the sense that for each $p \in \Sigma_d$ there is a neighbourhood U_p in M such that its intersection with Σ_d contains only p. On the other hand Σ_c will be a submanifold of M. Both sets are connected by the property that the geodesic null cone of each $p \in \Sigma_d$ is completely contained in Σ.

First simple consequences of the defined decomposition are summarised as follows.

Proposition 5.1.1. *Consider $\Sigma = \Sigma_c \cup \Sigma_d$ as defined above and let $p \in \Sigma_d$. Then*

(i) *there are coordinates $(U(p), \varphi)$ such that $\pm\sigma = -\left(\varphi^0\right)^2 + \left(\varphi^1\right)^2 + \cdots + \left(\varphi^{n-1}\right)^2$.*

(ii) *p is an isolated point of Σ_d.*

(iii) *Σ_c is an $(n-1)$-dimensional null submanifold of (M, g).*

Proof: By definition of Σ_d we have $\sigma(p) = 0$ and $d\sigma_p = 0$. Corollary 1.4.12(ii) then states $\rho(p) \neq 0$ such that by Equation (1.4.12) at p the Hessian $\operatorname{Hess}\sigma_p = -\rho(p)g_p$ is a multiple of the metric. In particular p is a non-degenerate critical point of σ. Hence the index of σ in p is

$n-1$ for $\rho(p) > 0$ and 1 for $\rho(p) < 0$. By applying the Morse Lemma 1.5.2 we get coordinates $(U, \varphi = (\varphi^0, \dots, \varphi^{n-1}))$ in a neighbourhood U of p such that $\varphi(p) = 0$ and

$$\sigma = s\left(\left(\varphi^0\right)^2 - \left(\varphi^1\right)^2 - \dots - \left(\varphi^{n-1}\right)^2\right)$$

holds on U, where $s := \operatorname{sgn}(\rho)$ is the sign of ρ.

As Σ_d is a subset of the set of critical points of σ, by Corollary 1.5.3 each of its elements and in particular p is an isolated point of Σ_d.

The last claim then is a consequence of the regular value theorem. First we observe that due to point (ii), $M \setminus \Sigma_d$ is an open subset and n-dimensional submanifold of M. Now consider the restriction $\sigma : M \setminus \Sigma_d \to \mathbb{R}$. Since we removed all critical points in $\sigma^{-1}(0)$, 0 is a regular value of the restricted map and therefore its preimage Σ_c is an $(n-1)$-dimensional submanifold of $M \setminus \Sigma_d$ and hence of M. It is isotropic, since $\operatorname{grad} \sigma$ is a non-vanishing null vector on $T\Sigma_c$. ■

Lemma 5.1.2. *Let $\gamma : I \to \Sigma$ be a smooth curve with non-vanishing tangent vector $\dot\gamma$. If it passes a point in Σ_d then its tangent vector is a null vector at that point, in particular*

$$\gamma(t_0) \in \Sigma_d \;\Rightarrow\; g_{\gamma(t_0)}\left(\dot\gamma(t_0), \dot\gamma(t_0)\right) = 0.$$

Proof: Since γ is a smooth curve in Σ, we have $\dot\gamma(t) \in T_p\Sigma_c$ as long as $\gamma(t) \in \Sigma_c$. On the other hand $\operatorname{grad} \sigma_{\gamma(t)} = 0$ for $\gamma(t) \in \Sigma_d$. Hence $d\sigma_{\gamma(t)}(\dot\gamma) = g_{\gamma(t)}(\operatorname{grad} \sigma_{\gamma(t)}, \dot\gamma(t)) = 0$ for all $t \in I$. We define the function f by $f(t) := g_{\gamma(t)}(\operatorname{grad} \sigma_{\gamma(t)}, \dot\gamma(t))$. On the one hand it vanishes identically. On the other hand it is a composition of smooth maps and hence can be differentiated such that

$$\begin{aligned} 0 &\equiv \dot f(t) \\ &= g\left(\nabla_{\dot\gamma}\dot\gamma, \operatorname{grad}\sigma\right)(t) + g\left(\dot\gamma, \nabla_{\dot\gamma}\operatorname{grad}\sigma\right)(t) \\ &= g\left(\nabla_{\dot\gamma}\dot\gamma, \operatorname{grad}\sigma\right)(t) + \left(\operatorname{Hess}\sigma(\dot\gamma,\dot\gamma)\right)(t). \end{aligned}$$

By using Corollary 1.4.12(i) we get $(\rho g(\dot\gamma,\dot\gamma))(t) = -g\left(\nabla_{\dot\gamma}\dot\gamma, \operatorname{grad}\sigma\right)(t)$. Non-vanishing of ρ at $t = t_0$ (Corollary 1.4.12(ii)) then gives

$$g_{\gamma(t_0)}\left(\dot\gamma(t_0), \dot\gamma(t_0)\right) = -\frac{1}{\rho\left(\gamma(t_0)\right)} g\left(\nabla_{\dot\gamma}\dot\gamma, 0\right)(t_0) = 0.$$

Hence $\dot\gamma(t_0)$ is a null vector as claimed. ■

Next we will point out some properties of the coordinates found above.

Lemma 5.1.3. *Let $p \in \Sigma_d$, (U, φ) the chart found in Proposition 5.1.1 and $\{\partial_i\}$ the associated local coordinate frame. Then it holds for all $i \neq 0$ and $\mu \neq \nu$*

$$\begin{aligned} g_p(\partial_0, \partial_0) &= -g_p(\partial_i, \partial_i) \\ g_p(\partial_\mu, \partial_\nu) &= 0 \end{aligned}$$

such that up to a constant, in p the metric is diagonal in those coordinates and $\{\kappa\partial_i\}$ is a orthonormal frame for some $\kappa \in \mathbb{R}$.

As g is a Lorentzian metric and $\dim M \geq 3$, we get ∂_0 to be a timelike vector in p, while ∂_i are spacelike vectors.

Proof: Consider $(X^0, \dots, X^{n-1}) \in \mathbb{R}^n$ with $\left(X^0\right)^2 = 1 = \left(X^1\right)^2 + \dots + \left(X^{n-1}\right)^2$ and $X := X^0 e_0 + \dots + X^{n-1}e_{n-1}$, where $\{e_0, \dots, e_{n-1}\}$ is the standard basis in $\mathbb{R}^n$. We define the smooth curve $\gamma(t) := \varphi^{-1}(tX)$ for small values of $|t|$. Then $\dot\gamma(t) = \left(X^0\partial_0 + \dots + X^{n-1}\partial_{n-1}\right)_{\gamma(t)}$ is its non-vanishing tangent vector. Moreover, we have $\gamma(0) = p$ and $\sigma \circ \gamma(t) = 0$. Hence γ maps to Σ. From Lemma 5.1.2 we conclude γ to have null tangent vector at $t = 0$, in particular $g_p(\dot\gamma(0), \dot\gamma(0)) = 0$.

We may now consider special choices for X to derive the claimed equations for the tangent vectors ∂_i. First let $X = e_0 \pm e_i$ with $i \neq 0$, then we get

$$\begin{aligned} 0 &= g_p\left(\partial_0 + \partial_i, \partial_0 + \partial_i\right) - g_p\left(\partial_0 - \partial_i, \partial_0 - \partial_i\right) \\ &= 4 g_p\left(\partial_0, \partial_i\right) \end{aligned}$$

and

$$\begin{aligned} 0 &= g_p\left(\partial_0 + \partial_i, \partial_0 + \partial_i\right) + g_p\left(\partial_0 - \partial_i, \partial_0 - \partial_i\right) \\ &= 2 g_p\left(\partial_0, \partial_0\right) + 2 g_p\left(\partial_i, \partial_i\right), \end{aligned}$$

which proves the first line and parts of the second one. Now let $X = e_0 + \frac{1}{\sqrt{2}}\left(e_i + e_j\right)$ with $i \neq j$ and $i, j \neq 0$. This choice gives

$$\begin{aligned} 0 &= g_p\left(\partial_0 + \frac{1}{\sqrt{2}}\left(\partial_i + \partial_j\right), \partial_0 + \frac{1}{\sqrt{2}}\left(\partial_i + \partial_j\right)\right) \\ &= g_p\left(\partial_0, \partial_0\right) + \frac{1}{2}\left(g_p\left(\partial_i, \partial_i\right) + g_p\left(\partial_j, \partial_j\right)\right) + g_p\left(\partial_i, \partial_j\right) \\ &= g_p\left(\partial_i, \partial_j\right), \end{aligned}$$

where in each step we used the results of the first calculation. This proves the remaining equations. ■

5.1.2. Null Cone for Points in Σ_d

The aim of this section is to show local equality of Σ and the geodesic null cones in a neighbourhood of points $p \in \Sigma_d$. In this section, it will be useful to interpret the gradient vector field $\operatorname{grad}\sigma$ as a field on Σ. To shorten the notation, $\operatorname{grad}\sigma|_\Sigma : \Sigma \to TM$ will just be denoted $\operatorname{grad}\sigma$ if there is no risk of confusion.

We will first specify the structure of Σ that is induced by the Morse lemma in a neighbourhood of points in Σ_d.

Lemma 5.1.4. *Let be $p \in \Sigma_d$, then there is a neighbourhood $\mathcal{U}$ of p such that $\Sigma \cap \mathcal{U}$ is a cone quadric over p.*

Proof: Following the proof of Lemma 1.2.5, there is a neighbourhood $\mathcal{U}$ of p and a diffeomorphism $\varphi : \mathcal{U} \to \mathbb{R}^n$ such that $\Sigma \cap \mathcal{U}$ is the preimage of 0 with respect to $-\left(\varphi^0\right)^2 + \left(\varphi^1\right)^2 + \cdots + \left(\varphi^{n-1}\right)^2$. The sign does not matter here. Finally $\Sigma \cap \mathcal{U}$ is homeomorphic to the $(n-1)$ dimensional double cone in $\mathbb{R}^n$. ■

Lemma 5.1.5. *The restricted map* $\operatorname{grad}\sigma|_{\Sigma_c} : \Sigma_c \to TM$ *has the following properties.*

(i) $\operatorname{grad}\sigma_p$ *is an element of $T_p\Sigma_c \subset T_pM$ for each $p \in \Sigma_c$ and hence a tangent vector field of Σ_c. Consequently one has* $\operatorname{grad}\sigma|_{\Sigma_c} \in \mathfrak{X}(\Sigma_c)$.

(ii) *The $(1,1)$ tensor $\nabla \operatorname{grad}\sigma$ fulfils on Σ*

$$\nabla_X \operatorname{grad}\sigma =|_\Sigma - \rho X. \tag{5.1}$$

(iii) *Let $X \in \mathfrak{X}(M)$ be a vector field that is tangent to Σ_c, i.e. $g\left(X, \operatorname{grad}\sigma\right) = 0$ on Σ_c. Then $\nabla_{\operatorname{grad}\sigma} X$ also is tangent to Σ_c. In other words*

$$\forall p \in \Sigma_c : g_p\left(X_p, \operatorname{grad}\sigma_p\right) = 0 \Rightarrow \forall p \in \Sigma_c : g_p\left(\nabla_{\operatorname{grad}\sigma} X_p, \operatorname{grad}\sigma_p\right) = 0.$$

Proof: Consider $p \in \Sigma_c$. For (i) it suffices to show $\operatorname{grad}\sigma_p \in T_p\Sigma_c$. Let $\gamma : I \to \Sigma_c$ be an arbitrary smooth curve with $\gamma(0) = 0$ and $\dot\gamma(0) = X \in T_p\Sigma_c$. Then $\sigma \circ \gamma \equiv 0$ along γ and therefore

$$0 = \left.\frac{d}{dt}\right|_{t=0} \sigma \circ \gamma = g_p\left(\operatorname{grad}\sigma_p, X_p\right) = d\sigma_p\left(X_p\right). \tag{5.2}$$

Since in addition we have $g(\operatorname{grad}\sigma_p, \operatorname{grad}\sigma_p) = 0$, $\operatorname{grad}\sigma_p$ must be an element of $T_p\Sigma_c$. Assuming $\operatorname{grad}\sigma_p$ would be in the algebraic complement then any basis of $T_p\Sigma$ could be completed by $\operatorname{grad}\sigma_p$ to a basis of T_pM, which on the other hand would be annihilated by $d\sigma_p$. This would contradict p to be an element of Σ_c.

For the second claim we may use that by (1.5) and (1.6) on an almost Einstein structure (M, g, σ) we find

$$\nabla \operatorname{grad}\sigma + \sigma \mathrm{P}^\sharp + \rho \operatorname{id} = 0.$$

Consequently on Σ this gives $\nabla \operatorname{grad}\sigma = \rho \operatorname{id}$.

By (i) $\operatorname{grad}\sigma$ is tangent to Σ. Hence the last claim is consequence of the following calculation

$$\begin{aligned} g\left(\nabla_{\operatorname{grad}\sigma} X, \operatorname{grad}\sigma\right) &=|_\Sigma \operatorname{grad}\sigma\left(g\left(X, \operatorname{grad}\sigma\right)\right) - g\left(X, \nabla_{\operatorname{grad}\sigma}\operatorname{grad}\sigma\right) \\ &\overset{(ii)}{=}\Big|_\Sigma \operatorname{grad}\sigma(0) + \rho g(X, \operatorname{grad}\sigma) \\ &=|_\Sigma 0. \end{aligned}$$

■

As $\operatorname{grad}\sigma$ identifies the isotropic direction on Σ_c in Lorentzian signature this implies that every tangent null vector in $T\Sigma_c$ will be a multiple of $\operatorname{grad}\sigma$. This has the following consequences.

Lemma 5.1.6. *Let $p \in \Sigma_c$ be a point, $X \in T_p\Sigma$ a null vector and $\gamma : I \to \Sigma_c$ a smooth curve with non-vanishing null tangent vector $\dot\gamma$. Then*

(i) *there is a constant $\kappa \in \mathbb{R}$ such that $X = \kappa \cdot \operatorname{grad}\sigma_p$.*

(ii) *γ is a null pregeodesic.*

(iii) *any integral curve of $\operatorname{grad}\sigma$ starting on Σ_c stays within Σ_c for all values of its parameter.*

(iv) *any integral curve of $\operatorname{grad}\sigma$ on Σ_c is a null pregeodesic.*

(v) *γ can be reparametrised to an integral curve of $\operatorname{grad}\sigma$.*

Proof: The statements will be proven separately.

(i) The first claim is a consequence of Σ_c being a null submanifold. Assume $X \in T_p\Sigma$ to be a null vector in $T\Sigma_c$. From (5.2) we get $g(\operatorname{grad}\sigma, X) = 0$ such that $\operatorname{span}\{\operatorname{grad}\sigma, X\}$ is a totally isotropic vector subspace of T_pM. As the signature of M is Lorentzian, the subspace has dimension one and the claim follows.

(ii) From part (i) we conclude that there is a smooth map $c : I \to \mathbb{R} \setminus \{0\}$ such that $\dot\gamma(t) = c(t) \operatorname{grad}\sigma_{\gamma(t)}$. We now show γ to fulfil Equation (1.78)

$$\begin{aligned} \left(\nabla_{\dot\gamma}\dot\gamma\right)(t) &= \left(\nabla_{\dot\gamma} c \cdot \operatorname{grad}\sigma\right)(t) \\ &= \dot c(t) \operatorname{grad}\sigma_\gamma(t) + c(t)\left(\nabla_{\dot\gamma}\operatorname{grad}\sigma\right)(t) \\ &\overset{(5.1)}{=} \left(\frac{\dot c(t)}{c(t)} - c(t)\rho \circ \gamma(t)\right)\dot\gamma(t). \end{aligned}$$

Consequently it is a pregeodesic.

(iii) As stated before the restriction of $\operatorname{grad}\sigma$ to Σ_c is a vector field in $\mathfrak{X}(\Sigma_c)$. Therefore the differential equation $\dot{\eta}(t) = \operatorname{grad}\sigma_{\eta(t)}$ on Σ_c with initial data $\eta(0) = p \in \Sigma_c$ and $\dot{\eta}(0) = \operatorname{grad}\sigma_p \in T_p\Sigma_c$ has a unique solution $\eta : (-\epsilon, \epsilon) \to \Sigma_c$. The claim follows directly.

(iv) Let $\eta : I \to \Sigma_c$ be an integral curve of $\operatorname{grad}\sigma$. Then by definition $\dot{\eta}(t) = \operatorname{grad}\sigma_{\eta(t)}$ is a non-vanishing null vector for all $t \in I$. Hence η satisfies the requirements of (ii) and so is a pregeodesic.

(v) Let $\eta : J \to \Sigma_c$ be the smooth null curve with non-vanishing tangent vector, parametrised by the interval $J = (t_0, t_1)$. By (i) there is a smooth non-vanishing map $f : J \to \mathbb{R}$ such that $\dot{\eta}(t) = f(t)\operatorname{grad}\sigma_{\eta(t)}$. Reparametrisation with a function $h : \tilde{J} := (s_0, s_1) \to J$ leads to the required differential equation:

$$\begin{aligned}(\eta \circ h)' &= h' \cdot f \circ h \cdot \operatorname{grad}\sigma_{\gamma\circ h} \\ &\overset{!}{=} \operatorname{grad}\sigma_{\gamma\circ h}\end{aligned}$$

where we denote the derivative with respect to the parameter on $\tilde{J}$ with a prime. The resulting ordinary differential equation $h' \cdot f \circ h = 1$ is implicitly solved by separation of variables to

$$\int_{h(s_0)}^{h(s)} \frac{dx}{f(x)} = s - s_0.$$

The integral on the left-hand side is strictly monotonic as a function of $h(s)$ due to non-vanishing of f. Hence a solution $h(s)$ to this equation exists. It is strictly monotonic too and has an inverse. As a result η can be reparametrised to an integral curve of $\operatorname{grad}\sigma$ on Σ_c.

■

Lemma 5.1.7. *Let $\gamma : (\alpha, \beta) \to \Sigma_c$ be a maximal integral curve of* $\operatorname{grad}\sigma$. *If the limit $\lim_{t\to\alpha}\gamma(t)$ or $\lim_{t\to\beta}\gamma(t)$ exists, then it is an element of Σ_d*

Proof: The gradient vector field is non-vanishing on Σ_c. Hence for each point in Σ_c we can find a chart neighbourhood such that at least one component of $\operatorname{grad}\sigma$ is nowhere vanishing on this neighbourhood. By Lemma 1.1.16 every maximal integral curve entering that neighbourhood will leave it within finite values of the parameter. Therefore no point in Σ_c can be the limit of a maximal integral curve. Since Σ is a closed subset of M and $\gamma(t) \in \Sigma$ for all $t \in (\alpha, \beta)$, the limit must be in Σ and by the previous considerations in $\Sigma \setminus \Sigma_c = \Sigma_d$ if existent. This proves the claim. ■

Lemma 5.1.8. *Every point $p \in \Sigma_d$ is an attractor or repeller of* $\operatorname{grad}\sigma$ *in M.*

Proof: Let be (U_p, x) a normal coordinate chart at p, with $x = (x^0, \dots, x^{n-1}) : U_p \to \mathbb{R}^n$. The intention in this proof is to make use of the theorem of Poincaré and Lyapunov 1.1.17.

In coordinates we have

$$\operatorname{Hess}^g \sigma = \sum_{i,j=0}^{n-1} (\partial_i\partial_j\sigma)\, dx^i \otimes dx^j - \sum_{i,j,k=0}^{n-1} \left(\Gamma^k_{ij}\partial_k\sigma\right) dx^i \otimes dx^j$$

for the Hessian of σ and

$$\begin{aligned}\operatorname{grad}\sigma &= \sum_{i,j=0}^{n-1} g^{ij}(\partial_j\sigma)\partial_i \\ &= \sum_{i=0}^{n-1} \operatorname{grad}\sigma^i\partial_i\end{aligned}$$

for the gradient. Here g^{ij} are the components of the inverse matrix of g at $y \in U_p$. The coordinate representation $\widetilde{\operatorname{grad}\sigma} := (\operatorname{grad}\sigma^0 \circ x^{-1}, \ldots, \operatorname{grad}\sigma^{n-1} \circ x^{-1})$ is a smooth map from $x(U_p) \subset \mathbb{R}^n$ to $\mathbb{R}^n$. We will drop the tilde in the following considerations and identify the components of $\operatorname{grad}\sigma$ with this representation. In normal coordinates the Christoffel symbols Γ^k_{ij} vanish at p and hence

$$\operatorname{Hess}\sigma_p = \sum_{i,j=0}^{n-1} \left(\partial_i \partial_j \sigma\right)(p) \left(dx^i \otimes dx^j\right)_p .$$

Corollary 1.4.12(i) then gives

$$\left(\partial_i \partial_j \sigma\right)(p) = -\rho(p) g_{ij}(p). \tag{5.3}$$

We now calculate the coordinate Jacobian $\operatorname{grad}\sigma'$. Using Einstein notation its components are

$$\begin{aligned}(\operatorname{grad}\sigma')^i_k &= \partial_k g^{ij} \partial_j \sigma \\ &= (\partial_k g^{ij})\partial_j\sigma + g(ij\partial_k\partial_j\sigma.\end{aligned}$$

By requirements of the lemma, p is a critical point with $\operatorname{grad}\sigma(p) = 0$ and hence the first term vanishes at p. Combining the result with Equation (5.3) gives

$$\begin{aligned}(\operatorname{grad}\sigma')^i_k(p) &= -\rho(p)\sum_j g^{ij}(p) g_{kj}(p) \\ &= -\rho(p)\delta^i_k,\end{aligned} \tag{5.4}$$

where δ^i_k are the components of the identity matrix. By Corollary 1.4.12(ii) this implies that $\operatorname{grad}\sigma'$ does not vanish at p and moreover is positive or negative definite depending on the sign of ρ. Now the theorem of Poincaré and Lyapunov can be applied, which gives the claim. In particular p is an attractor for $\rho(p) < 0$ and an repeller for $\rho(p) > 0$. ∎

Corollary 5.1.9. *Let be $p \in \Sigma_d$. Then with respect to $\operatorname{grad}\sigma$, there is an attracting or repelling neighbourhood U_p of p that also is a normal neighbourhood of p. In particular $\mathfrak{U} := \exp_p^{-1}(U_p)$ is a convex neighbourhood of 0 in T_pM.*

Proof: Due to Lemma 5.1.8, p is an attractor or repeller. Following the calculations therein, Equation (5.4) gives $(\operatorname{grad}\sigma')^i_k(p) = -\rho(p)\delta^i_k$ in normal coordinates (U, φ). Therefore $\operatorname{grad}\sigma'(0)$ clearly is positive or negative definite and hence using Proposition 1.1.18 there is an open ball $B_r(0) \subset \varphi(U)$, which is an attracting or repelling neighbourhood of 0 with respect to $\varphi_* \operatorname{grad}\sigma$. Consequently $U_p := \varphi^{-1}(B_r(0))$ is an attracting or repelling neighbourhood. The ball is taken with respect to the canonical vector space metric in $\mathbb{R}^n$.

Moreover, $B_r(0)$ corresponds to an open ball in T_pM in a suitable basis as we started with normal coordinates φ. Specifically there is a basis $\{e_i\}$ of T_pM such that $Y \in B_r(0)$ is mapped to $\exp^{-1} \circ \varphi^{-1}(Y) = Y^0 e_0 + \cdots + Y^{n-1} e_{n-1}$ in $\mathfrak{U}$. ∎

This corollary gives rise to the following definition.

Definition 5.1.10. Let be $p \in M$ an attractor or repeller of $X \in \mathfrak{X}(M)$. A neighbourhood $\mathfrak{U} \subset T_pM$ of 0 is called *stable neighbourhood* with respect to X at p, if (i) $\exp_p(\mathfrak{U})$ is a normal neighbourhood of p and (ii) $\exp_p(\mathfrak{U})$ is an attracting or repelling neighbourhood of p with respect to X.

Lemma 5.1.11. *For $p \in \Sigma_d$ there are convex sets $\mathfrak{U} \subset \mathfrak{K} \subset \tilde{\mathfrak{U}} \subset T_pM$ containing the origin 0 such that $\mathfrak{K}$ is compact and $\mathfrak{U}$, $\tilde{\mathfrak{U}}$ are stable neighbourhoods with respect to $\operatorname{grad}\sigma$ at p.*

A proof can be found in the appendix.

Remark. As a matter of fact $\tilde{\mathfrak{U}}$ can be chosen such that its image under $\exp_p$ is a subset of some convex neighbourhood of p. First choose some convex neighbourhood $\mathcal{V}$ of p. Then $\varphi\left(\exp_p^{-1}(\mathcal{V})\right)$ is well defined and open in $\mathbb{R}^n$. Hence using the construction of the proof above r can be scaled down such that the ball $B_r(0)$ is a subset of $\exp_p^{-1}(\mathcal{V})$ and still is attracting or repelling. The last claim holds, since every ball inside an attracting ball sharing its centre is attracting or repelling itself.

Proposition 5.1.12. *For $p \in \Sigma_d$ the singularity set Σ locally coincides with the geodesic null cone in p. In particular there is a neighbourhood U_p of p such that $U_p \cap \Sigma = \mathcal{C}_p(U_p)$.*

Proof: First using Lemma 5.1.11 we choose convex subsets $\mathfrak{U} \subset \mathfrak{K} \subset \tilde{\mathfrak{U}} \subset T_pM$ such that $\mathfrak{K}$ is compact and $\mathfrak{U}, \tilde{\mathfrak{U}}$ are stable neighbourhoods of the origin. We will in advance require $\tilde{\mathfrak{U}}$ to be small enough such that $\exp_p(\tilde{\mathfrak{U}})$ is subset of some convex neighbourhood of p. We then define the following sets

$$
\begin{aligned}
K &:= \exp_p(\mathfrak{K}) & \tilde{\mathcal{U}} &:= \exp_p(\tilde{\mathfrak{U}}) \\
\mathfrak{C} &:= \left(\mathfrak{C}_pM \cup \{0\}\right) \cap \mathfrak{K} & \tilde{\mathfrak{C}} &:= \mathfrak{C}_pM \cap \tilde{\mathfrak{U}} \\
\mathfrak{S} &:= \exp_p^{-1}(\Sigma \cap K) \subset \mathfrak{K} & \tilde{\mathfrak{S}} &:= \exp_p^{-1}\left(\Sigma_c \cap \tilde{\mathcal{U}}\right) \subset \tilde{\mathfrak{U}}
\end{aligned}
$$

It suffices to show that $\mathfrak{S}$ is an open and closed subset of $\mathfrak{C}$. We remark that $p \in \mathfrak{C} \cap \mathfrak{S}$ by definition such that it suffices to show that $\mathfrak{S} \setminus \{0\}$ is open and closed in $\mathfrak{C} \setminus \{0\}$.

Since Σ is a closed set, we see immediately that $\mathfrak{S}$ is closed. We will show it to be a subset of $\mathfrak{C}$. Consider $x \in \Sigma \cap K \subset \exp_p\left(\tilde{\mathfrak{U}}\right)$. Since $\tilde{\mathfrak{U}}$ is stable, there is a null integral curve $\gamma : I \to \Sigma_c$ with $\gamma(0) = x$ and either $I = (-\infty, 0]$ or $I = [0, \infty)$ such that $\gamma(t) \to p$ for $t \to \pm\infty$. Due to Lemma 5.1.6(ii) γ is a pregeodesic and can be reparametrised to a segment of a null geodesic $\eta : [0, t_0) \to \Sigma_c$ with $\eta(0) = x$ and $\eta \to p$ for $t \to t_0$. Since $\exp_p\left(\tilde{\mathfrak{U}}\right)$ is subset of a convex neighbourhood of p and there is no geodesic ending inside M (see Lemma 1.2.4), η can be extended to the value t_0 by $\eta(t_0) = p$. Reparametrising η to a radial geodesic in p gives $x = \exp_p\left(-t_0\dot{\eta}(t_0)\right)$ for the null vector $\dot{\eta}(t_0)$. Therefore we have $-t_0\dot{\eta}(t_0) \in \mathfrak{C}_pM \cap \mathfrak{K} \subset \mathfrak{C}$ and we conclude that $\mathfrak{S}$ is a closed subset of $\mathfrak{C}$ and consequently $\mathfrak{S} \setminus \{0\}$ is a closed subset of $\mathfrak{C} \setminus \{0\}$.

Using the same arguments as above we get $\tilde{\mathfrak{S}}$ to be a subset of $\tilde{\mathfrak{C}}$. At this point we summarise the results gained so far as follows

$$
\mathfrak{S} \setminus \{0\} \subset_{\text{closed}} \mathfrak{C} \setminus \{0\} \qquad\qquad \tilde{\mathfrak{S}} \subset \tilde{\mathfrak{C}}.
$$

Now we will show $\mathfrak{S} \setminus \{0\}$ to be open in $\mathfrak{C} \setminus \{0\}$. First we assume $x \in \Sigma_c \cap \tilde{\mathcal{U}}$ and hence by the previous considerations $\exp_p^{-1}(x) \in \tilde{\mathfrak{S}} \subset \tilde{\mathfrak{C}}$. It suffices to show that there is a neighbourhood U of x such that

$$
\exp_p^{-1}\left(U \cap (\Sigma \cap K)\right) = \exp_p^{-1}(U) \cap \mathfrak{C},
$$

since then $\exp_p^{-1}(U) \cap \exp_p^{-1}\left((\Sigma \cap K)\right) = \exp_p^{-1}(U) \cap \mathfrak{C}$ and therefore the set on the left-hand side contains $\exp_p^{-1}(x)$ and is open in $\mathfrak{C} \setminus \{0\}$. According to the first part of the proof we already have

$$
\exp_p^{-1}\left(U \cap (\Sigma \cap K)\right) = \exp_p^{-1}(U) \cap \mathfrak{S} \subset \exp_p^{-1}(U) \cap \mathfrak{C}
$$

independent of the choice of U.

For a concrete choice we assert Σ_c to be a submanifold of M. Therefore we can choose a chart (U, φ) of x such that $\varphi(U \cap \Sigma_c) = V \times \{0\} \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^{n-1}$ is an open set. We assume U to be small enough such that $U \subset \exp_p\left(\tilde{\mathfrak{U}} \setminus \{0\}\right)$. In particular $(\exp_p^{-1}(U), \tilde{\varphi})$ with $\tilde{\varphi} := \varphi \circ \exp_p$ is a chart for a neighbourhood of $\exp_p^{-1}(x)$ in T_pM. Moreover we have

$$\begin{aligned}\tilde{\varphi}\left(\exp_p^{-1}(U) \cap \tilde{\mathfrak{C}}\right) &\supset \tilde{\varphi}\left(\exp_p^{-1}(U) \cap \tilde{\mathfrak{S}}\right) \\ &= \tilde{\varphi}\left(\exp_p^{-1}\left(U \cap \Sigma_c\right)\right) \\ &= \varphi\left(U \cap \Sigma_c\right) \\ &= V \times \{0\}.\end{aligned}$$

$\mathfrak{C} = \mathfrak{C}_pM \cap \tilde{\mathfrak{U}}$ topologically is an open subset of $\mathfrak{C}_pM$ and hence an $(n-1)$-dimensional submanifold. Consequently $(\exp_p^{-1}(U) \cap \tilde{\mathfrak{S}}, \tilde{\varphi})$ must be a chart for a neighbourhood of $\exp_p^{-1}(x)$ in $\tilde{\mathfrak{C}}$. In particular $\exp_p^{-1}(U) \cap \tilde{\mathfrak{S}}$ is a open neighbourhood of $\exp_p^{-1}(x)$ in $\tilde{\mathfrak{C}}$. By arbitrariness of $x \in \Sigma_c \cap \tilde{\mathcal{U}}$ we get that $\tilde{\mathfrak{S}}$ is open in $\tilde{\mathfrak{C}}$. Hence $\mathfrak{S} \setminus \{0\} = \tilde{\mathfrak{S}} \cap \mathfrak{K}$ is an open subset of $\mathfrak{C} \setminus \{0\} = \tilde{\mathfrak{C}} \cap \mathfrak{K}$. Therefore we have $\mathfrak{C} \setminus \{0\} = \mathfrak{S} \setminus \{0\}$ and hence

$$\mathfrak{C} = \mathfrak{S}.$$

We now choose $U_p := \exp_p(\mathfrak{U})$. Then since $\mathfrak{U} \subset \mathfrak{K}$, we get

$$\mathfrak{C}_pM \cap \mathfrak{U} = \exp_p^{-1}\left(\Sigma_c \cap U_p\right). \tag{5.5}$$

Proposition 1.2.8 (Equation (1.85)) states that in a normal neighbourhood the geodesic null cone $\mathcal{C}_p(U_p)$ in p is the image of the tangent null cone under the exponential map. Hence

$$\begin{aligned}\mathcal{C}_p(U_p) &= \exp_p(\mathfrak{C}_pM \cap \mathfrak{U}) \\ &\overset{(5.5)}{=} \Sigma_c \cap U_p.\end{aligned}$$

This proves the claim. ■

5.1.3. Flow of grad σ

In this section we will give a more detailed analysis of the level sets of σ in a neighbourhood of vertices $p \in \Sigma_d$. This will help to identify some properties of the flow of grad σ on Σ_c. We already named the level set of level zero by $\Sigma = \sigma^{-1}(0)$. The remaining level sets will be denoted

$$\Sigma^s := \sigma^{-1}(s).$$

for $s \in \mathbb{R}$. In a more abstract context we will also use the notation Σ^σ for the level sets. In the following we will not distinguish between the future and past time cone or between the future and past null cone. Nevertheless the inside and outside of a local null cone will be specified as follows.

Definition 5.1.13. Let $p \in \Sigma_d$ be a vertex of Σ and $\mathcal{U}$ a normal neighbourhood with $\mathfrak{U} \subset T_pM$ being its preimage. Then the *inside* $I(p,\mathcal{U})$ of the local geodesic null cone is the set generated by all timelike vectors in $\mathfrak{U}$, i.e. $I(p,\mathcal{U}) = \exp_p\left(\mathfrak{T}_pM \cap \mathfrak{U}\right)$. The *outside* is the set generated by all spacelike vectors in $\mathfrak{U}$, i.e. $O(p,\mathcal{U}) := \exp_p\left(\mathfrak{U} \setminus \mathfrak{K}_pM\right)$.

We point out that since $\mathcal{U}$ is required to be a normal neighbourhood, we have the disjoint union

$$\mathcal{U} = I(p,\mathcal{U}) \cup O(p,\mathcal{U}) \cup \mathcal{C}_p(\mathcal{U}). \tag{5.6}$$

Lemma 5.1.14. *Let $p \in \Sigma_d$ be a vertex of Σ. Then there is a normal neighbourhood $\mathcal{U}$ of p such that the inside and outside of the geodesic null cone in p are characterised by the causality character of* grad σ *in the following sense:*

$$\begin{aligned}I(p,\mathcal{U}) &= \{x \in \mathcal{U} \mid \operatorname{grad}\sigma_x \textit{ timelike}\} \\ O(p,\mathcal{U}) &= \{x \in \mathcal{U} \mid \operatorname{grad}\sigma_x \textit{ spacelike}\}.\end{aligned}$$

Proof: Due to Corollary 1.4.12, there is a neighbourhood of p such that $\rho(x) \neq 0$ for all x in that neighbourhood. Now consider the Morse chart $(\mathcal{V}, \varphi)$ with $\varphi(p) = 0$, as found in Proposition 5.1.1. We will require $\mathcal{V}$ to be small enough such that ρ is non-vanishing within it. In those coordinates we have $\sigma = s\left((\varphi^0)^2 - (\varphi^1)^2 - \cdots - (\varphi^{n-1})^2\right)$, where $s = \operatorname{sgn}(\rho(p))$. From $S(g,\sigma) = 0$ we conclude

$$\begin{aligned} g(\operatorname{grad}\sigma, \operatorname{grad}\sigma) &= -2\rho\sigma \\ &= |\rho|\left(-\left(\varphi^0\right)^2 + \left(\varphi^1\right)^2 + \cdots + \left(\varphi^{n-1}\right)^2\right) \end{aligned}$$

for all $x \in \mathcal{V}$. Further we notice that for all $q \in \mathcal{V}$ we have $q \in \Sigma \overset{\rho\neq 0}{\Longleftrightarrow} \rho(q)\sigma(q) = 0 \Leftrightarrow g_q(\operatorname{grad}\sigma_q, \operatorname{grad}\sigma_q) = 0$ and hence by Proposition 5.1.12 there is a normal neighbourhood $\mathcal{U} \subset \mathcal{V}$ such that for all $q \in \mathcal{U}$ we have

$$q \in \mathcal{C}_p(\mathcal{U}) \Longleftrightarrow g_q(\operatorname{grad}\sigma_q, \operatorname{grad}\sigma_q) = 0.$$

We conclude that $\operatorname{grad}\sigma$ must not change its causal character within the connected components of $I(p,\mathcal{U})$ and $O(p,\mathcal{U})$. For dimension $n > 2$ the outside $O(p,\mathcal{U})$ is connected, while $I(p,\mathcal{U})$ has two connected components. Hence we just have to calculate the causal character of $\operatorname{grad}\sigma$ for one point in each connected component.

Lemma 5.1.3 states that there is a $\kappa \in \mathbb{R}^+$ such that the canonic frame $\{\kappa\partial_\mu\}$ with respect to the coordinates is orthonormal in p and $[\partial_0]_p$ is a timelike vector. Hence the curve $\gamma(t) := \varphi^{-1}(\pm t \cdot (1,0,\ldots,0))$ is a timelike curve for t small enough and hence maps to $I(p,\mathcal{U})$. The sign in the definition of γ determines the connected component of $I(p,\mathcal{U})$, to which it maps. In addition we have $g_{\gamma(t)}(\operatorname{grad}\sigma, \operatorname{grad}\sigma) = -|\rho \circ \gamma(t)|t^2 < 0$ independently of the sign in the definition of γ such that $\operatorname{grad}\sigma$ is timelike for all $q \in I(p,\mathcal{U})$. We conclude

$$q \in I(p,\mathcal{U}) \Longrightarrow g_q(\operatorname{grad}\sigma_q, \operatorname{grad}\sigma_q) < 0.$$

We now define the curve $\gamma(t) := \varphi^{-1}(\pm t \cdot (0,1,0,\ldots,0))$ such that $g_{\gamma(t)}(\operatorname{grad}\sigma, \operatorname{grad}\sigma) = |\rho \circ \gamma(t)|t^2 > 0$ and hence by the decomposition (5.6) $\gamma(t)$ must be a curve outside the geodesic null cone in p. Moreover, $\operatorname{grad}\sigma$ must be spacelike for all $g \in O(p,\mathcal{U})$, which together with the last result completes the proof. ∎

Corollary 5.1.15. *Let be $p \in \Sigma_d$ a vertex of Σ. Then locally the inside and outside of the geodesic null cone can be distinguished by the sign of $\rho\sigma$ as follows.*

$$\begin{aligned} I(p,\mathcal{U}) &= \{x \in \mathcal{U} \mid \rho(x)\sigma(x) > 0\} \\ O(p,\mathcal{U}) &= \{x \in \mathcal{U} \mid \rho(x)\sigma(x) < 0\}. \end{aligned}$$

The corollary is a consequence of the equality $g(\operatorname{grad}\sigma, \operatorname{grad}\sigma) = -2\rho\sigma$ and the last lemma.

Proposition 5.1.16. *Let $p \in \Sigma_d$ be a vertex. Then there is a normal neighbourhood $\mathcal{U}$ of p such that*

(i) all level sets Σ^s inside the local null cone in p are spacelike hypersurfaces, i.e. all connected components of $\Sigma^s \cap I(p,\mathcal{U})$ have a timelike normal vector.

(ii) all level sets Σ^s outside the local null cone are Lorentzian hypersurfaces, i.e. on any connected component of $\Sigma^s \cap I(p,\mathcal{U})$, the normal vector field is spacelike.

Proof: Following the proof of Proposition 5.1.1 we assert p to be isolated in the set of critical points of σ. Hence there is a neighbourhood $\mathcal{U}$ of p such that $\operatorname{grad}\sigma_q \neq 0$ for all $q \in \mathcal{U} \setminus \{p\}$. Due to Corollary 5.1.15 $\mathcal{U}$ can be reduced to an open normal neighbourhood such that inside and outside of the local geodesic cone $\mathcal{C}_p(\mathcal{U})$ are characterised by the sign of $\rho\sigma$.

Now let Σ^s be the level set to the level s. Then we have $\operatorname{grad}\sigma_q \neq 0$ for all $q \in \Sigma^s \cap \mathcal{U}$. Hence s is a regular value of σ if restricted to $\mathcal{U}$ and hence $\Sigma^s \cap \mathcal{U}$ and all its connected components are $n-1$ dimensional submanifolds. For $s \neq 0$ we have $g(\operatorname{grad}\sigma, \operatorname{grad}\sigma) \neq 0$ and therefore $\operatorname{grad}\sigma$

is transversal to Σ^s and orthogonal to its tangent space. By Lemma 5.1.14 grad σ is timelike inside the local null cone and hence the hypersurfaces $\Sigma^s \cap \mathcal{U}$ inside the null cone are spacelike. Outside the null cone grad σ is spacelike and therefore the hypersurfaces $\Sigma^s \cap \mathcal{U}$ outside the cone are Lorentzian hypersurfaces. ■

Lemma 5.1.17. *Let (M, g) be geodesically null complete. If $\Delta\sigma$ is bounded on Σ_c, then every maximal integral curve $\gamma : (\alpha, \beta) \to \Sigma_c$ of $\operatorname{grad} \sigma$ that starts on Σ_c is complete.*

Proof: For every $p \in \Sigma_d$ let $\mathcal{U}_p$ be an attracting or repelling neighbourhood as constructed in Corollary 5.1.9 and define

$$\mathcal{U} := \bigcup_{p \in \Sigma_d} \mathcal{U}_p.$$

If $\gamma(t) \in \mathcal{U}$ for a $t \in (\alpha, \beta)$ then by definition of attractors or repellers either $\alpha = -\infty$ or $\beta = \infty$. Hence γ is at least complete in one direction.

Now consider the case where $\gamma(t) \notin \mathcal{U}$ for all $t \in (\alpha_0, \beta)$ for some $\alpha_0 > \alpha$. By Lemma 5.1.6(iv) we find γ to be a null pregeodesic and hence by Lemma 1.2.3 there is a reparametrisation $h : (h_0, h_1) \to (\alpha_0, \beta)$ to a null geodesic $\eta := \gamma \circ h : (h_0, h_1) \to \Sigma_c$. Without loss of generality let be $h(h_1) = \beta$. We will show η to be inextendible to h_1 and therefore $h_1 = \infty$ due to the null completeness of (M, g).

Assume η to be extendible to the interval $(h_0, h_1]$ and $\eta(h_1) = p \in M$. Then p is in Σ since $\Sigma = \sigma^{-1}(0)$ is a closed subset of M. Moreover, as we required $\gamma(t) \notin \mathcal{U}$ the limit p must be in Σ_c and $\gamma(t) \to p$ and $\dot\gamma(t) \to \operatorname{grad} \sigma_p \neq 0$ for $t \to \beta$. Since $\operatorname{grad} \sigma_p \neq 0$ there is a coordinate neighbourhood (U, φ) of p such that for one component of $\operatorname{grad} \sigma$ in those coordinates we have $(\operatorname{grad} \phi)^k(x) > \delta$ for all $x \in U$. For any ball $B(0)$ in these coordinates, there is a t_0 such that $\varphi \circ \gamma(t) \in B$ for all $t > t_0$. If we require B to be small enough such that $B_{2\epsilon} \subset \varphi(U)$, we can apply Lemma 1.1.16 and hence there is a $t > t_0$ such that $\varphi \circ \gamma(t) \notin B$, which is a contradiction.

Hence η is inextendible to h_1 and we find η to be defined on the interval (h_0, ∞), in particular

$$\eta : (h_0, \infty) \to \Sigma_c.$$

Now we will show that the reparametrisation h in direction of $h_1 = \infty$ must be complete, i.e. $h(s) \to \infty$ for $s \to \infty$.

The reparametrisation constructed in Lemma 1.2.3 fulfils Equation (A.4)

$$s - h_0 = \int_{\alpha_0}^{h(s)} \exp[C(t)]dt$$

with $C(t) = \int_{t_0}^{t} c(x)dx$. In particular for the integral curve γ we have $c(x) = -\rho \circ \gamma(x) = -\frac{1}{n}(\Delta\sigma) \circ \gamma(x)$. The Laplacian of σ is bounded on Σ. Let be $\frac{|\Delta\sigma|}{n} \leq B < \infty$, then the right-hand side can be estimated to

$$\begin{aligned} s - h_0 &\leq \int_{\alpha_0}^{h(s)} \exp\left[B(t - t_0)\right] dt \\ &= \frac{\exp(-Bt_0)}{B} \left(\exp\left(Bh(s)\right) - \exp\left(B\alpha_0\right)\right). \end{aligned}$$

Hence $h(s) \to \infty$ for $s \to \infty$. Therefore the integral curve is defined on the interval (α, ∞).

The same arguments hold for the interval (α, β_0) for some $\beta_0 < \beta$. In particular either γ is repelled by a $p \in \Sigma_d$ and hence $\alpha = \infty$ or complete in that direction by changing the sign in the above arguments. ■

A consequence of the last lemma is the following proposition.

Proposition 5.1.18. *Assume that (M, g) is geodesically null complete and $\Delta\sigma$ is bounded on Σ_c, then the maximal flow $\Phi : \mathcal{U} \subset \mathbb{R} \times M \to M$ of $\operatorname{grad} \sigma$ is complete on Σ_c, i.e. the restriction $\Phi : \mathbb{R} \times \Sigma_c \to \Sigma_c$ is well defined on the hole interval $\mathbb{R}$ and*

$$\Phi(t, \Sigma_c) = \Sigma_c$$

for all $t \in \mathbb{R}$.

Remark. Following the last claims, the gradient vector field $\operatorname{grad}\sigma$ is complete if restricted to Σ_c and hence $\Phi : \mathbb{R} \times \Sigma_c \to \Sigma_c$ is a one parameter family of diffeomorphisms [O'N83, Lemma 1.54].

Lemma 5.1.19. *Let* $\Phi : \mathcal{D}^\Phi \subset \mathbb{R} \times \Sigma_c \to \Sigma_c$ *be the maximal flow of a vector field* $V \in \mathfrak{X}(\Sigma_c)$. *Furthermore let be* $S \subset \Sigma_c$ *a* $n-2)$*-dimensional submanifold of* Σ_c *with the following properties*

$$\begin{array}{lll} (*) & \forall t \neq 0, (t, S) \subset \mathcal{D}^\Phi & : \quad \Phi(t, S) \cap S = \emptyset \\ (**) & \forall x \in S & : \quad T_x\Sigma_c = T_x S \oplus \langle V(x) \rangle \end{array}$$

where $\langle V(x) \rangle$ *is the line in* $T_x M$ *spanned by* $V(x)$. *If* $f : U \subset \mathbb{R}^{n-2} \to S$ *is a diffeomorphism parametrising* S, *then the map* $\tilde{\Phi} := \Phi \circ (\mathrm{id}, f)$ *defined by*

$$\begin{array}{rccc} \tilde{\Phi} : & \mathbb{R} \times U \supset \mathcal{D}^{\tilde{\Phi}} & \to & \Phi\left(\mathcal{D}^\Phi\right) \\ & (t, \alpha) & \mapsto & \Phi(t, f(\alpha)) \end{array}$$

with $\mathcal{D}^{\tilde{\Phi}} := \left((\mathrm{id}, f)^{-1}(\mathbb{R} \times f(U)) \cap \mathcal{D}^\Phi\right)$ *is a diffeomorphism onto its image.*

Proof: Since f is a diffeomorphism, the map $\tilde{\Phi}$ is surjective by definition. For showing injectivity consider $\Phi(t_1, f(\alpha_1)) = \Phi(t_2, f(\alpha_2))$. Since Φ is the maximal flow, this means that $f(\alpha_1) = \Phi(0, f(\alpha_1)) = \Phi(t_2 - t_1, f(\alpha_2))$. From requirement $(*)$ we immediately get $t_2 - t_1 = 0$. Hence $\alpha_1 = \alpha_2$ as f is a diffeomorphism.

Now consider the differential of $\tilde{\Phi}$ in some arbitrary point $(t, \alpha) \in \mathcal{D}^{\tilde{\Phi}}$ evaluated on $(X_0, \mathbf{X}) \in \mathbb{R}^{n-1}$. Then we get

$$\begin{aligned} d\left[\Phi \circ (\mathrm{id}, f)\right]_{(t,\alpha)}(X_0, \mathbf{X}) &= d\Phi_{(t,f(\alpha))}(X_0, df_\alpha(\mathbf{x})) \\ &= X_0 V_{\Phi(t,f(\alpha))} + [d\Phi_t]_{f(\alpha)}(df_\alpha(\mathbf{X})), \end{aligned}$$

where $\Phi_t = \Phi(t, \dot{})$ is the t-th stage of Φ. As Φ_t is a local diffeomorphism and $df_\alpha(\mathbf{X})$ is transversal to $V_{f(\alpha)}$, also $[d\Phi_t]_{f(\alpha)}(df_\alpha(\mathbf{X}))$ is transversal to $V_{\Phi(t,f(\alpha))}$ provided $\mathbf{X} \neq 0$. Consequently $d\tilde{\Phi}_{(t,\alpha)}$ is an isomorphism for all $(t, \alpha) \in \mathcal{D}^{\tilde{\Phi}}$. ■

5.1.4. Conformal Compactifications

The definition of an almost Einstein structure (M, g, σ) does not necessarily coincide with a conformal compactification of $\left(M \setminus \Sigma, \sigma^{-2}g\right)$. The following section examines the properties of special almost Einstein structures that admit a conformal compactification. In particular we will require (M, g) to have a compact subset $\bar{M} \subset M$ that is the closure of a connected component of $M \setminus \Sigma$. We will next label the sets of interest.

Let (M, g, σ) be an almost Einstein structure. Then the set $M \setminus \Sigma$ may have several connected components. This will be labelled by i. Then fixing one component, we do the following notations

$$\begin{array}{ll} \mathring{M}_i & \text{fixed connected component of } M \setminus \Sigma \\ M_i & \text{connected component of } \mathring{M}_i \cup \Sigma_c \\ \bar{M}_i & \text{connected component of } M_i \cup \Sigma_d \\ \partial\mathring{M}_i & := \bar{M}_i \cap \Sigma. \end{array}$$

With this definition we have the inclusions $\mathring{M}_i \subset M_i \subset \bar{M}_i$. Provided $\mathring{M}_i$ is a conformally Einstein manifold with boundary, then $\partial\mathring{M}_i$ will be its conformal boundary.

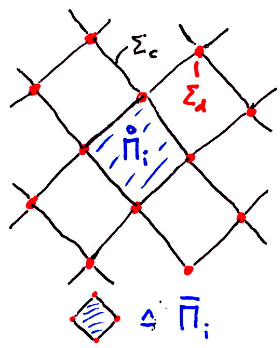

Figure 3.: schematics of a compactification

Definition 5.1.20. Let N be a manifold. A property that is defined for every $x \in N$ will be called *open* if it holds on an open subset of N.

Lemma 5.1.21. *Let be $p \in \Sigma_c \cap \partial^{top}\mathring{M}_i$ a point in the topological boundary of $\mathring{M}_i$. Then all points q in the same connected component of Σ_c also belong to the topological boundary of $\mathring{M}_i$.*

Proof: We will show that the properties $q \in \Sigma_c \cap \partial^{top}\mathring{M}_i$ and $q \notin \Sigma_c \cap \partial^{top}\mathring{M}_i$ are open properties in Σ_c. In particular for all $q \in \Sigma_c$ with $q \in \Sigma_c \cap \partial^{top}\mathring{M}_i$ there is a neighbourhood U of q such that for all $x \in U \cap \Sigma_c$ it holds $x \in \Sigma_c \cap \partial^{top}\mathring{M}_i$. The same can be said for $q \notin \Sigma_c \cap \partial^{top}\mathring{M}_i$.

First consider an arbitrary $q \in \Sigma_c$ such that $q \notin \Sigma_c \cap \partial^{top}\mathring{M}_i$. Since $\partial^{top}\mathring{M}_i$ is a closed set in M, there is an open neighbourhood $U \subset M$ of q such that $U \cap \partial^{top}\mathring{M}_i = \emptyset$. Hence q admits a neighbourhood $U \cap \Sigma_c$ open in Σ such that the property $x \notin \Sigma_c \cap \partial^{top}\mathring{M}_i$ holds for all $x \in U \cap \Sigma_c$. Hence it is an open property in Σ_c.

Now let $q \in \Sigma_c$ be arbitrary such that this time $q \in \Sigma_c \cap \partial^{top}\mathring{M}_i$. There is a coordinate chart $\varphi = (\varphi^1, \ldots, \varphi^n) : U \to \mathbb{R}^n$ for a neighbourhood of q such that $\varphi^n(x) = 0$ for all $x \in U \cap \Sigma_c$. Therefore we have $d\varphi^n_x(Y) = 0$ for all such x and $Y \in T_x\Sigma_c$ and hence $d\left(\varphi^1, \ldots, \varphi^{n-1}\right)_x(Y) \neq 0$ for all tangent vectors of Σ_c. We define $\tilde{\varphi} := \left(\varphi^1, \ldots, \varphi^{n-1}, \sigma\right) : U \to \mathbb{R}^n$. Since $d\sigma_x(Y) \neq 0$ for transversal vectors Y in $x \in \Sigma_c$, we find $\ker d\tilde{\varphi}_x = \{0\}$ for all $x \in U \cap \Sigma_c$ and in particular for $x = q$. By the inverse function theorem there is a neighbourhood $\tilde{U}$ of q such that

$$\tilde{\varphi} : \tilde{U} \to \mathbb{R}^n \tag{5.7}$$

is a coordinate chart and $\tilde{\varphi}(\tilde{U}) = B$ is an open ball. Since $q \in \partial^{top}\mathring{M}_i$ it holds that $\tilde{U} \cap \mathring{M}_i \neq \emptyset$. Consider $x \in \tilde{U} \cap \mathring{M}_i$ and without loss of generality $\sigma(x) > 0$, then by connectedness of the open ball we get $\tilde{\varphi}^{-1}\left(B \cap \left(\mathbb{R}^{n-1} \times (0, \infty)\right)\right) \subset \mathring{M}_i$ and hence $\tilde{U} \cap \Sigma_c = \tilde{\varphi}^{-1}\left(B \cap \left(\mathbb{R}^{n-1} \times \{0\}\right)\right) = \tilde{U} \cap \Sigma_c \subset \partial^{top}\mathring{M}_i$. Consequently $q \in \Sigma_c \cap \partial^{top}\mathring{M}_i$ is an open property too.

Now we have the disjoint union of sets $\Sigma_c = \Sigma_c^1 \cup \Sigma_c^2$ open in Σ_c where $\Sigma_c^1 := \{p \in \Sigma_c \mid p \in \partial^{top}\mathring{M}_i\}$ and $\Sigma_c^2 := \{p \in \Sigma_c \mid p \notin \partial^{top}\mathring{M}_i\}$. In particular the connected components of Σ_c must completely belong to one of those sets. ■

Lemma 5.1.22. *$\partial\mathring{M}_i$ coincides with the topological boundary $\partial^{top}\mathring{M}_i$ of $\mathring{M}_i$ in M.*

Proof: First, we will show the inclusion $\partial\mathring{M}_i \subset \partial^{top}\mathring{M}_i$. Consider $p \in \Sigma_c \cap \partial\mathring{M}_i$ and therefore $p \in M_i$. By definition, M_i is connected such that there is a curve $\gamma : [0,1] \to M_i$ with $\gamma(0) = p$ and $\gamma(1) =: x \in \mathring{M}_i$. We will consider γ to be a curve with values in $M_i \cup \Sigma$. Let be $t_0 := inf\,\{t \in [0,1] \mid \gamma(t) \in \Sigma_c\}$. Since Σ is a closed set and $\gamma(t)$ must not be in Σ_d, the limit $p_0 := \gamma(t_0)$ must be an element of Σ_c such that the restriction $\gamma : [0, t_0] \to \Sigma_c$ is completely within the connected component of Σ_c. For each of the neighbourhoods U_{p_0} of p_0 the intersection $U_{p_0} \cap \mathring{M}_i$ is non-empty, since due to the definition of t_0 there must be a $t' > t_0$ such that $\gamma(t') \in U_{p_0} \cap \mathring{M}_i$. Hence p_0 is an element of the topological boundary. By Lemma 5.1.21 the whole connected component of Σ_c and in particular $p = \gamma(0)$ are in the topological boundary. Therefore we get

$$M_i \subset \overline{\mathring{M}_i}. \tag{5.8}$$

Now consider $p \in \Sigma_d \cap \partial\mathring{M}_i$. By Proposition 5.1.1(ii), p is an isolated point in Σ_d. Therefore every neighbourhood U_p of p has non-vanishing intersection with M_i, since $\bar{M}_i$ is connected. Hence p is an element of the topological boundary $\partial^{top}M_i$, which is also a subset of $\partial^{top}\mathring{M}_i$, since $M_i \subset \overline{\mathring{M}_i}$ by Equation (5.8). This completes the first inclusion $\partial\mathring{M}_i \subset \partial^{top}\mathring{M}_i$.

Second we show that the backward inclusion $\partial^{top}\mathring{M}_i \subset \partial\mathring{M}_i$ holds. $\mathring{M}_i$ is a connected component of the open set $M \setminus \Sigma$. Now consider $p \in \partial^{top}\mathring{M}_i$, then p cannot be in the open set $M \setminus \Sigma$ and hence $p \in \Sigma$. First let $p \in \Sigma_c$ and consider $\tilde{\varphi}$ to be the coordinate chart in a neighbourhood of p that has been defined in Equation (5.7). Since $\tilde{U} \cap \mathring{M}_i \neq \emptyset$, there is a curve connecting p to a point in the interior $\tilde{U} \cap \mathring{M}_i$ such that σ is strictly positive along the curve. Consequently $p \in M_i$ and hence $p \in \partial\mathring{M}_i$. Now let $p \in \Sigma_d$. Then in Morse coordinates $M \setminus \Sigma$ consists of three connected open components. Since each of them is connected to the vertex by a path, p must be an element of $\bar{M}_i$ and hence of $\partial\mathring{M}_i$. This proves the backward inclusion. ■

The previous observations lead to the following proposition on the topology of the level sets in a neighbourhood of a compact conformal boundary Σ.

Proposition 5.1.23. *If $\partial\mathring{M}_i$ is compact and M geodesically null complete without boundary, then*

(i) The number of vertices in any connected component of $\Sigma \cap \partial\mathring{M}_i$ is finite.

(ii) There exists a pre-compact neighbourhood $\mathcal{U}$ of $\partial\mathring{M}_i$ in M such that $\mathring{M}_i$ can be written as

$$\mathring{M}_i \cap \mathcal{U} = \bigcup_{s\in(-\epsilon,\epsilon)} \left(\Sigma^s \cap \mathring{M}_i \cap \mathcal{U}\right)$$

with $\Sigma^s \cap \mathring{M}_i \cap \mathcal{U}$ being $(n-1)$-dimensional hypersurfaces not leaving $\mathcal{U}$, i.e. $\partial\mathcal{U} \cap \left(\Sigma^s \cap \mathring{M}_i\right) = \emptyset$. In particular there are no critical points of σ within $\mathring{M}_i \cap \mathcal{U}$.

(iii) The gradient vector field $\operatorname{grad}\sigma$ if restricted to $\partial\mathring{M}_i$ is complete.

Proof: The third claim is a consequence of Proposition 5.1.18. By the requirements $\partial\mathring{M}_i \subset \Sigma$ is compact and hence $\Delta\sigma$ is bounded on $\partial\mathring{M}_i \cap \Sigma_c$. By Lemma 5.1.21 $\partial\mathring{M}_i \cap \Sigma_c$ contains only connected parts of Σ_c as a whole and not just parts of it. Now applying Proposition 5.1.18 gives the claim.

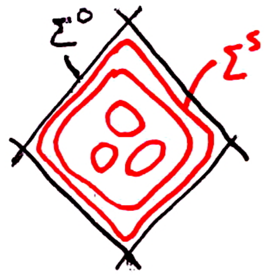

Figure 4.: topology near compact boundaries

The first claim is a consequence of the compactness of $\partial\mathring{M}_i$. Let be $p \in \Sigma_d \cap \partial\mathring{M}_i$ a vertex. By Proposition 5.1.1(ii), p is an isolated point such that it admits a neighbourhood U_p without any other vertices. For $p \in \Sigma_c \cap \partial\mathring{M}_i$ choose a neighbourhood U_p without any vertices in it. With these choices

$$\partial\mathring{M}_i \subset \bigcup_{p\in\partial\mathring{M}_i} U_p$$

is a covering and by compactness admits a finite subcovering. Since the neighbourhood of every vertex must be an element of that covering, their number consequently must be finite.

The proof for the second claim uses the same idea. First we observe that for each $p \in \Sigma$ there is a connected neighbourhood U_p such that $\operatorname{grad}\sigma \neq 0$ for all $q \in U_p \setminus \{p\}$. Moreover, we demand each neighbourhood to be such that it includes only one connected component of Σ and its closure $\overline{U_p}$ is supposed to be compact. Those neighbourhoods provide an open covering of $\partial\mathring{M}_i$ that must have a finite subcovering

$$U = \bigcup_{j\in\{1,..,m\}} U_{p_j}.$$

Its closure $\overline{U} = \bigcup_{i\in\{1,..,m\}} \overline{U_{p_i}}$ is compact and so is its boundary ∂U. By construction we have

$$\partial U \cap \partial\mathring{M}_i = \emptyset. \tag{5.9}$$

Without loss of generality assume $\mathring{M}_i$ to be a subset of $\sigma^{-1}((0,\infty))$. We will now prove by contradiction that there is an ϵ such that for all $0 < \delta < \epsilon$ we have $\Sigma^\delta \cap \bar{M}_i \cap \partial U = \emptyset$. So assume that for all $\epsilon > 0$ there exists a $\delta \in (0,\epsilon]$ such that $\Sigma^\delta \cap \bar{M}_i \cap \partial U \neq \emptyset$. Hence we can choose for every $\epsilon = \frac{1}{i}$ an element of that intersection

$$x_i \in \Sigma^{\frac{1}{i}} \cap \bar{M}_i \cap \partial U.$$

By the compactness of $\bar{U}$ there is a subsequence $\{x_{i_j}\}$ that converges to an $x \in \partial U$. By construction we have $\sigma(x_{i_j}) = \frac{1}{i_j}$ for that sequence. By continuity of σ we have $\sigma(x) = 0$ and hence $x \in \partial\mathring{M}_i$. This contradicts (5.9). The same argument holds for $\mathring{M}_i \subset \sigma^{-1}((-\infty,0))$, so the first assumption really is no loss of generality.

As a consequence there exists an $\epsilon > 0$ such that $\Sigma^\delta \cap \bar{M}_i \cap \partial U = \emptyset$ for all $0 < |\delta| \leq \epsilon$. The connected parts of Σ^δ within $\mathring{M}_i \cap U$ are hypersurfaces since by construction $d\sigma \neq 0$ on U. We define

$$\mathcal{U} := U \cap \left(\sigma^{-1}((-\epsilon,\epsilon)) \cup \bar{M}_i^c\right)$$

where $\bar{M}_i^c$ is the complement of $\bar{M}_i$ in M. the set $\mathcal{U}$ by definition is open, contains $\partial\mathring{M}_i$ and the connected parts of Σ^δ within $\mathring{M}_i \cap U$ for $\delta \in (-\epsilon, \epsilon)$. Hence we get

$$\mathcal{U} \cap \mathring{M}_i = \mathcal{U} \cap \sigma^{-1}((-\epsilon, \epsilon)) \cap \mathring{M}_i$$

and the second claim follows directly by replacing $\sigma^{-1}((-\epsilon, \epsilon))$ with the union of level sets. ■

Remark. By restricting σ to the open set $\mathring{M}_i \cap \mathcal{U}$, we are left with a smooth function $\sigma : \mathring{M}_i \cup \mathcal{U} \to \mathbb{R}$. $\sigma^{-1}([\alpha_0, \alpha_1])$ are compact sets for $0 < \alpha_i < \epsilon$ if σ is positive on $\mathring{M}_i$ and $-\epsilon < \alpha_i < 0$ else. Hence by Theorem 1.5.1 the sets $\sigma^{-1}(\infty, \alpha_0)$ and $\sigma^{-1}(\infty, \alpha_1)$ are diffeomorphic and the former set is a deformation retract of the latter. Since $d\sigma$ is non-singular and closed in the constructed neighbourhood of $\partial\mathring{M}_i$, the level sets of σ represent a 1-dimensional foliation of $\mathring{M}_i$ in a neighbourhood of its boundary.

5.1.5. Focal Points on Σ

The vertices of Σ are special in the sense that they prevent Σ from being a smooth null hypersurface. They actually have more interesting properties. The content of the next section will be to show that the only focal points along null geodesics in Σ with respect to any $(n-2)$-dimensional Riemannian submanifold $N \subset \Sigma$ are exactly the vertices. This result would be a direct corollary of Lemma 5.1.6 if only focal points in Σ were considered. But a priori we will allow variations of the geodesic that do not belong to Σ.

Lemma 5.1.24. *Let (M, g, σ) be an almost Einstein structure, $\gamma : I \to \Sigma$ a null geodesic on Σ and $f : I \to \mathbb{R}$ a smooth map such that $f(t)\dot{\gamma}(t) = \operatorname{grad}\sigma(t)$. Moreover, let $J \in \mathfrak{X}(\gamma)$ be a vector field along γ. Then*

$$f \cdot \mathrm{R}^g(J, \dot{\gamma})\dot{\gamma} = g(\dot{\gamma}, \operatorname{grad}\rho)J - g(\dot{\gamma}, J)\operatorname{grad}\rho - g(J, \operatorname{grad}\rho)\dot{\gamma}.$$

Proof: Using $\operatorname{Hess}\sigma^\sharp = -\sigma \mathrm{P}^\sharp - \rho\,\mathrm{id}$ on almost Einstein structures we get at $\gamma(t)$

$$\begin{aligned}
f \cdot \mathrm{R}^g(J, \dot{\gamma})\dot{\gamma} &= \mathrm{R}^g(J, \dot{\gamma})\operatorname{grad}\sigma \\
&= \nabla_J \nabla_{\dot{\gamma}} \operatorname{grad}\sigma - \nabla_{\dot{\gamma}} \nabla_J \operatorname{grad}\sigma - \nabla_{[J,\dot{\gamma}]}\operatorname{grad}\sigma \\
&\overset{(1.5)}{=} \nabla_J \left(\operatorname{Hess}\sigma^\sharp(\dot{\gamma})\right) - \nabla_{\dot{\gamma}} \left(\operatorname{Hess}\sigma^\sharp(J)\right) - \operatorname{Hess}\sigma^\sharp([J, \dot{\gamma}]) \\
&= \nabla_{\dot{\gamma}} \left(\sigma \mathrm{P}^\sharp(J) + \rho J\right) - \nabla_J \left(\sigma \mathrm{P}^\sharp(\dot{\gamma}) + \rho\dot{\gamma}\right) + \sigma \mathrm{P}^\sharp([J, \dot{\gamma}]) + \rho[J, \dot{\gamma}] \\
&= (\nabla_{\dot{\gamma}}\sigma)\,\mathrm{P}^\sharp(J) + (\nabla_{\dot{\gamma}}\rho)\,J - (\nabla_J\sigma)\,\mathrm{P}^\sharp(\dot{\gamma}) - (\nabla_J\rho)\,\dot{\gamma} \\
&\quad + \sigma\left(\nabla_{\dot{\gamma}}\left(\mathrm{P}^\sharp(J)\right) - \nabla_J\left(\mathrm{P}^\sharp(\dot{\gamma})\right) + \mathrm{P}^\sharp([J, \dot{\gamma}])\right) \\
&\quad + \rho\left(\nabla_{\dot{\gamma}}J - \nabla_J\dot{\gamma} + [J, \dot{\gamma}]\right) \\
&= g(\dot{\gamma}, \operatorname{grad}\sigma)\,\mathrm{P}^\sharp(J) + g(\dot{\gamma}, \operatorname{grad}\rho)\,J - g(J, \dot{\gamma})\,\mathrm{P}^\sharp(\operatorname{grad}\sigma) - g(J, \operatorname{grad}\rho)\,\dot{\gamma} \\
&\quad + \sigma\left(\left(\nabla_{\dot{\gamma}}\mathrm{P}^\sharp\right)(J) - \left(\nabla_J\mathrm{P}^\sharp\right)(\dot{\gamma})\right).
\end{aligned}$$

For the transformation leading to the last line we used $g(X, \operatorname{grad}\sigma)T(\dot{\gamma}) = g(X, \dot{\gamma})T(\operatorname{grad}\sigma)$, due to the requirements on $\dot{\gamma}$. Now taking into account that for almost Einstein structures $d\rho = \mathrm{P}^\sharp(\operatorname{grad}\sigma)$ and that on Σ it holds $g(\operatorname{grad}\sigma, \dot{\gamma}) = 0$, we get the claimed result. ■

Corollary 5.1.25. *With the requirements of the last lemma J is a Jacobi field along γ if and only if*

$$f \cdot \nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}J = g(\dot{\gamma}, \operatorname{grad}\rho)J - g(\dot{\gamma}, J)\operatorname{grad}\rho - g(J, \operatorname{grad}\rho)\dot{\gamma}. \tag{5.10}$$

Proof: First consider J to be a Jacobi field, i.e. $\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}J = \mathrm{R}^g(J, \dot{\gamma})\dot{\gamma}$, then the claim follows from the last lemma. Conversely consider J to solve (5.10). By the last lemma we have $f \cdot \nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}J = f \cdot \mathrm{R}^g(J, \dot{\gamma})\dot{\gamma}$ and hence the Jacobi equation is fulfilled where $f \neq 0$. By Proposition 5.1.1 $\operatorname{grad}\sigma$ vanishes only on Σ at isolated points $p \in \Sigma_d$. Since $\dot{\gamma}$ is nowhere vanishing, we conclude from $f(t)\dot{\gamma}(t) = \operatorname{grad}\sigma_{\gamma(t)}$ along Σ that f may only be zero for isolated $t \in I$. The Jacobi equation then holds on a dense subset of I and hence by smoothness of J and γ all over the interval. ■

Lemma 5.1.26. *Let (M, g, σ) be an almost Einstein structure, $\gamma : I \to \Sigma_c$ a null geodesic and J an Jacobi field on γ such that for some $t_0 \in I$ it satisfies*

$$J(t_0) = 0 \qquad\qquad \nabla_{\dot\gamma} J(t_0) = \alpha\dot\gamma(t_0)$$

with $\alpha \in \mathbb{R}$. Then J is given explicitly by

$$J(t) = (t - t_0)\alpha\dot\gamma(t)$$

for all $t \in I$

Proof: The proof can be carried out by direct calculation. Consider $\tilde J = (t - t_0)\alpha\dot\gamma(t)$. Since γ is a geodesic, this yields $\nabla_{\dot\gamma}\tilde J = \alpha\dot\gamma$ and $\nabla_{\dot\gamma}\nabla_{\dot\gamma}\tilde J \equiv 0 \equiv \mathrm{R}^g(\tilde J, \dot\gamma)\dot\gamma$. Hence $\tilde J$ is a Jacobi field with initial data $\tilde J(t_0) = 0$ and $\dot{\tilde J}(t_0) = \alpha\dot\gamma(t_0)$. By uniqueness of Jacobi fields for given initial data, J and $\tilde J$ coincide. ∎

Lemma 5.1.27. *Let (M, g, σ) be an almost Einstein structure, $N \subset \Sigma$ an $(n-2)$-dimensional spacelike submanifold of Σ and $\gamma : I \to \Sigma$ a null geodesic starting at N, i.e. $\gamma(0) \in N$. Let J now be an N-Jacobi field on γ with $J(t_0) = 0$ for some $t_0 \neq 0$, then J and its covariant derivative $\nabla_{\dot\gamma} J$ along γ are tangent to Σ for all $t \in I$, i.e. $J(t), \nabla_{\dot\gamma} J \in T_{\gamma(t)}\Sigma$ for all $\gamma(t) \in \Sigma_c$.*

In particular, since a null vector tangent to Σ is perpendicular to any spacelike submanifold of Σ, γ is a null geodesic normal to N and therefore the requirements of the lemma imply $\gamma(t_0)$ to be a focal point of N. The proof is based on an idea of the proof for [O'N83, Lemma 8.7], namely considering the map $g_\gamma(J_\gamma, \dot\gamma) : I \to \mathbb{R}$.

Proof: First we observe that for each $p \in N$ we have the direct sum decomposition $T_p\Sigma = T_pN \oplus \langle \operatorname{grad}\sigma_p\rangle$. Since $\operatorname{grad}\sigma$ is normal to the tangent space of Σ_c it is normal to T_pN.

Now let $J \in \mathfrak{X}(\gamma)$ be a N-Jacobi field on the null geodesic γ with focal point at $\gamma(t_0)$, i.e. $J(t_0) = 0$. This gives

$$\begin{aligned}\frac{d^2}{dt^2} g(J, \dot\gamma) &= \nabla_{\dot\gamma}\nabla_{\dot\gamma} g(J, \dot\gamma)\\ &= g(\nabla_{\dot\gamma}\nabla_{\dot\gamma} J, \dot\gamma)\\ &= g(\mathrm{R}^g(J, \dot\gamma)\dot\gamma, \dot\gamma)\\ &\equiv 0\end{aligned}$$

As a result $g(J, \dot\gamma)$ is a linear function $\kappa(t) = \alpha + \beta \cdot t$ along γ. At $\gamma(0)$, J is tangent to N while $\dot\gamma$ is normal to N and we find $\kappa(0) = \alpha = 0$. On the other hand at $\gamma(t_0)$ the requirement $J(t_0) = 0$ gives $\kappa(t_0) = \beta \cdot t_0 = 0$ and hence $\beta = 0$. Hence $g(J, \dot\gamma)$ and $\frac{d}{dt} g(J, \dot\gamma) = g(\nabla_{\dot\gamma} J, \dot\gamma)$ are trivial maps along γ, which is equivalent to $J, \nabla_{\dot\gamma} J \in T_{\gamma(t)}\Sigma$ for all $t \in I$. ∎

Corollary 5.1.28. *Let (M, g, σ) be an almost Einstein structure, $N \subset \Sigma$ an $(n-2)$-dimensional spacelike submanifold of Σ and $\gamma : I \to \Sigma$ a null geodesic normal to N, in particular $\gamma(0) \in N$. If $\gamma(t_0)$ is an element of Σ_c and a focal point of N and if J is the related N-Jacobi field with $J(t_0) = 0$, then $\frac{\nabla}{dt} J(t_0)$ is tangent to γ in t_0, i.e.*

$$(\nabla_{\dot\gamma} J)(t_0) = \alpha\dot\gamma(t_0)$$

for some $\alpha \in \mathbb{R}$.

Proof: Let $0 \in H \subset \mathbb{R}$ be a small interval. First we will construct a geodesic variation $\delta\gamma : I \times H \to M$ normal to N such that J is its variation vector field, i.e.

$$\begin{aligned}\delta\gamma(t, 0) &= \gamma(t) & \qquad \delta\gamma(0, H) &\subset N\\ (\partial_s\delta\gamma)(t, 0) &= J(t) & \qquad (\partial_t\delta\gamma)(0, s) &\in T_{\delta\gamma(0,s)}N^\perp.\end{aligned}$$

Throughout the proof we will use the following short notation

$$\begin{aligned}\partial_s\partial_t\delta\gamma &:= \frac{\nabla}{ds}\partial_t\gamma & \qquad \alpha_t(s) &:= \delta\gamma(t, s)\\ \partial_t\partial_s\delta\gamma &:= \frac{\nabla}{dt}\partial_s\gamma & \qquad \alpha(s) &:= \alpha_0(s)\\ \dot J &:= \nabla_{\dot\gamma} J = \frac{\nabla}{dt} J. & &\end{aligned}$$

Derivatives with respect to t will be denoted with a dot, while derivatives with respect to s will be denoted with a prime. From Lemma 5.1.27 we get $\dot{J}(t) \in T_{\gamma(t)}\Sigma$ for all $t \in I$, where $\dot{J}(0) = \pi^N(\dot{J}(0)) + \kappa \operatorname{grad} \sigma_{\gamma(0)}$ for some $\kappa \in \mathbb{R}$, in particular $\pi^\perp(\dot{J}(0)) = \kappa \operatorname{grad} \sigma_{\gamma(0)}$. In addition, since J is an N-Jacobi field, we have $J(0) \in T_{\gamma(0)}N$. Now assume without loss of generality $\dot{\gamma}(0) = \operatorname{grad} \sigma_{\gamma(0)}$ and consider $\alpha : H \to N$ to be an arbitrary curve such that

$$\alpha(0) = \gamma(0) \qquad\qquad \alpha'(0) = J(0) \in T_{\gamma(0)}N.$$

We now define a vector field Z on α with values in the normal bundle over N by

$$\begin{array}{rccc} Z: & H & \to & TN^\perp \\ & s & \mapsto & (1+\kappa s)\operatorname{grad}\sigma_{\alpha(s)} \end{array}$$

If the interval H is sufficiently small and if α is well behaved, we will use the equivalent notation $Z(s) = Z_{\alpha(s)}$. By Lemma 5.1.5 we have $\nabla_{\alpha'} \operatorname{grad} \sigma = -\rho\alpha'$ on Σ and hence the projection to the normal bundle gives $\pi^\perp(\nabla_{\alpha'} \operatorname{grad}\sigma) \equiv 0$ along α such that

$$\begin{aligned} Z'(0) &= (\nabla_{\alpha'} Z)(0) \\ &= (\nabla_{\alpha'} \operatorname{grad}\sigma)(0) - \pi^\perp((\nabla_{\alpha'} \operatorname{grad}\sigma)(0)) + \kappa \operatorname{grad}\sigma_{\alpha(0)} \\ &= \tilde{\mathrm{II}}(\alpha'(0), \operatorname{grad}\sigma_{\alpha(0)}) + \pi^\perp(\dot{J}(0)) \\ &= \tilde{\mathrm{II}}(J(0), \dot{\gamma}(0)) + \pi^\perp(\dot{J}(0)) \\ &= \pi^N(\dot{J}(0)) + \pi^\perp(\dot{J}(0)) \\ &= \dot{J}(0). \end{aligned}$$

See Equation (1.80) for the definition of $\tilde{\mathrm{II}}$. Now we can define the desired geodesic variation $\delta\gamma$ via the normal exponential map $\exp : TN^\perp \to M$ on the normal bundle of N by:

$$\begin{array}{rccc} \delta\gamma: & I \times H & \to & M \\ & (t,s) & \mapsto & \exp_{\alpha(s)}(t \cdot Z(s)). \end{array}$$

Since we don't require M to be complete we may have to choose H sufficiently small such that $t \cdot Z(s)$ is in the domain of $\exp$. By definition $\delta\gamma(\cdot, s)$ are geodesics for fixed s and since $\partial_t \delta\gamma(0,s) = Z(s) \propto \operatorname{grad}\sigma_{\alpha(s)}$, they are normal to N such that $\delta\gamma$ clearly is a geodesic variation of null geodesics normal to N. Moreover, we have

$$\begin{aligned} \partial_t \partial_s \delta\gamma(0,0) &= \partial_s \partial_t \delta\gamma(0,0) \\ &= \partial_s Z(0) \\ &= \dot{J}(0). \end{aligned}$$

Therefore the Jacobi field defined by $\partial_s \delta\gamma(\cdot, 0)$ coincides with J at $t = 0$ and hence on the whole interval I due to uniqueness of Jacobi fields.

The second step is to derive $\dot{J}$ at $\gamma(t_0) = \delta\gamma(t_0, 0)$. We recall that at the focal point $\delta\gamma(t_0, 0)$ we have

$$\alpha'_{t_0}(0) = \partial_s \delta\gamma(t_0, 0) = J(t_0) = 0. \tag{5.11}$$

We observe that for fixed $s \in H$ the map $\delta\gamma(\cdot, s) : I \to \Sigma_c$ is a null geodesic. Now consider $t \in (t_0 - \epsilon, t_0]$ to be close enough to t_0 such that $\operatorname{grad}\sigma_{\gamma(t)} \neq 0$. If necessary shrink the interval H such that $\operatorname{grad}\sigma$ does not vanish along the variation $\delta\gamma$, i.e. $\operatorname{grad}\sigma_{\delta\gamma(t,s)} \neq 0$ for all $(t,s) \in (t_0 - \epsilon] \times H$. Then the tangent vector $\partial_t \delta\gamma(t,s)$ is collinear to $\operatorname{grad}\sigma$ and hence there is a smooth map $\eta : (t_0 - \epsilon, t_0] \times H \to \mathbb{R}$ such that

$$\partial_t \delta\gamma(t', s) = \eta(t', s) \operatorname{grad}\sigma_{\delta\gamma(t',s)}.$$

Now we derive $\dot{J}(t_0)$ as follows

$$\dot{J}(t_0) \quad = \quad \partial_t \partial_s \delta\gamma(t_0, 0)$$

$$
\begin{aligned}
&= \partial_s \partial_t \delta\gamma(t_0, 0) \\
&= \left(\nabla_{\alpha'_t} \eta(t_0, \cdot) \operatorname{grad}\sigma\right)(0) \\
&\overset{(5.1)}{=} \eta'(t_0, 0) \operatorname{grad}\sigma_{\delta\gamma(t,0)} - \rho\eta(t_0, 0)\alpha'_{t_0}(0) \\
&\overset{(5.11)}{=} \eta'(t_0, 0) \operatorname{grad}\sigma_{\delta\gamma(t_0,0)},
\end{aligned}
$$

which proves the proposition since $\operatorname{grad}\sigma_{\delta\gamma(t_0,0)} = \operatorname{grad}\sigma_{\gamma(t_0)} \propto \dot{\gamma}(t_0)$ ■

Proposition 5.1.29. *Let (M, g, σ) be an almost Einstein structure, $N \subset \Sigma_c$ an $(n-2)$-dimensional spacelike submanifold and $\gamma : [0, t_0] \to \Sigma$ a geodesic normal to N. If $\gamma(t_0)$ is a focal point of N with respect to γ then it is a vertex of Σ, i.e.*

$$\gamma(t_0) \textit{ focal point of } N \;\Rightarrow\; \gamma(t_0) \in \Sigma_d$$

Proof: We will show that a point in Σ_c cannot be a focal point of N with respect to γ. Consider $\gamma(t_0) \in \Sigma_c$ to be a focal point and J an N-Jacobi field on γ. Then by Corollary 5.1.28, $\dot{J}(t_0) = \alpha\dot{\gamma}(t_0)$. Lemma 5.1.26 then implies J to be of the form $J(t) = (t - t_0)\alpha\dot{\gamma}(t)$. Since $J(0) \in T_{\gamma(0)}N$, it must vanish at $t = 0$ such that $\alpha = 0$. But then $J \equiv 0$ contradicts J to be a non-vanishing vector field. Hence $\gamma(t_0) \in \Sigma_d$. ■

Let $p \in \Sigma_d$ be a vertex of Σ. We will now construct coordinates for a neighbourhood of a segment of a radial null geodesic $\eta : t \mapsto \exp_p(tu) \in \Sigma$. Proposition 1.2.8 guarantees $u \in \mathfrak{C}_pM$ in the first place. The geodesic does not have to be complete but we will consider it to be well defined for $t \in (-\epsilon, 1+\epsilon) =: I$ for some $\epsilon > 0$. Moreover, we require u to be sufficiently close to the origin of T_pM such that $\exp_p(tu) \in \Sigma_c$ for all $t \in I \setminus \{0\}$. In particular p is the only vertex along the null geodesic segment γ.

First we will fix some notation. We define

$$\mathfrak{U} \subset T_pM$$

to be a convex neighbourhood of the origin in T_pM. It is mapped by the exponential map to

$$\mathcal{U} := \exp_p(\mathfrak{U}).$$

Moreover, we will require $\mathfrak{U}$ to be small enough such that p is the only critical point of $\operatorname{grad}\sigma$ in $\mathcal{U}$, i.e.

$$\operatorname{grad}_{\exp_p(X)} = 0 \;\Leftrightarrow\; X = 0 \text{ for } X \in \mathfrak{U}^{1}.$$

Now let

$$u \in \mathfrak{C}_pM \setminus \mathfrak{U}$$

be a null vector outside $\mathfrak{U}$. The domain of $\exp_p$ is star-shaped with respect to the origin, hence there is an $s > 1$ and $\tilde{u} \in \mathfrak{U}$ such that

$$u = s\tilde{u}.$$

We now choose a neighbourhood $\mathfrak{U}_{\tilde{u}} \subset \mathfrak{U} \setminus \{0\}$ of $\tilde{u}$ not containing the origin and define the following objects

$$
\begin{aligned}
p &= \exp_p(0) & \widetilde{\mathfrak{U}} &:= \mathfrak{U}_{\tilde{u}} \cap \mathfrak{C}_pM \\
x &:= \exp_p(\tilde{u}) & \widetilde{\mathcal{U}} &:= \exp_p\left(\widetilde{\mathfrak{U}}\right) \\
q &:= \exp_p(u).
\end{aligned}
$$

Since $\widetilde{\mathfrak{U}}$ does not contain the origin, it is an $(n-1)$-dimensional submanifold of T_pM and a neighbourhood of $\tilde{u}$ within the null cone $\mathfrak{C}_pM$. The restriction $\exp_p : \widetilde{\mathfrak{U}} \to \widetilde{\mathcal{U}} \subset \Sigma_c$ then is a diffeomorphism.

1 This choice is possible due to Proposition 5.1.1

We assert that $\operatorname{grad}\sigma_{\eta(t)}$ is non-vanishing and tangent to $\eta(t)$ along the radial null geodesic $\eta : I \to \Sigma_c \cup \{p\}$ for $t \neq 0$. Moreover, we have $\eta\left(s^{-1}\right) = \exp_p(\tilde{u}) = x$ and $\eta(1) = \exp_p(u) = q$. Hence $\eta : [s^{-1}, 1] \to \Sigma_c$ is a null curve from x to q with non-vanishing tangent vector. By Lemma 5.1.6(iiv) it can be reparametrised to an integral curve of $\operatorname{grad}\sigma$. Expressing the integral curve in x via the flow of $\operatorname{grad}\sigma$, namely $\Phi(\cdot, x)$ by the previous considerations there has to be a $t_0 \in \mathbb{R}$ such that $\Phi(t_0, x) = q$. Without loss of generality we assume it to be positive, which is equivalent to assume p to be a repeller of $\operatorname{grad}\sigma$. We will point out the changes that will have to be made to adapt the following construction to a negative t_0.

We will now construct and define a special orthonormal frame of the tangent space in T_pM adapted to the null cone $\mathfrak{C}_pM$. Consider an orthonormal frame $\{e_i\}$ of T_pM such that $g(e_i, e_i) = 1$ for $i \in \{1, \dots, n-1\}$ and $g(e_0, e_0) = -1$. Moreover, it can be chosen such that $\tilde{u} = e_1 + e_0$[2]. The null cone in p can be written as

$$\mathfrak{C}_pM = \left\{ X = X^0 e_0 + X^0 \mathbf{e} \;\middle|\; X^0 \in \mathbb{R}, \mathbf{e} \in S^{n-2} \right\}$$

where we identify $\mathbf{e}$ with an element in S^{n-2}, since on the null cone it holds that $X^0\mathbf{e} = X^1 e_1 + \dots + X^{n-1} e_{n-1}$ with $\sum_{i=1}^{n} \left(X^i\right)^2 = \left(X^0\right)^2$ as a necessary and sufficient condition for null vectors in an orthonormal frame. This definition implies that

$$g_p(e_0 + e_1, e_i) = 0 \quad \forall i \geq 2,$$

such that $\{e_0 + e_1, e_i\}_{i \geq 2}$ generates the tangent space $T_{\tilde{u}}\left(\mathfrak{C}_pM\right) \subset T_{\tilde{u}}\left(T_pM\right) \simeq T_pM$ of the null cone in $\tilde{u}$. Consequently, we can choose coordinates on the sphere and consider $\mathbf{e}$ as coordinate function

$$\mathbf{e} : \mathcal{V} \subset \mathbb{R}^{n-2} \to T_pM \tag{5.12}$$

with the identification just made. We will require the coordinates to be such that[3]

$$\mathbf{e}(0) = e_1 \qquad\qquad (\partial_i \mathbf{e})(0) = e_{i+1}.$$

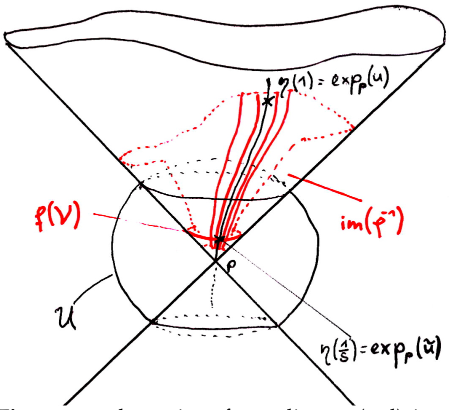

Figure 5.: schematics of coordinates (red) in a neighbourhood of null geodesics on Σ_c

Altogether the last considerations can be summarised as follows

Lemma 5.1.30. *Using the notation above, let $p \in \Sigma_d$ and $\eta : (0-\delta, 1+\delta) \to \Sigma_c \cup \{p\}$ a null geodesic of the form $\eta(t) = \exp_p(tu)$ for $u \in \mathfrak{C}_pM$. Then there exists a coordinate neighbourhood $\left(\mathrm{im}(\varphi^{-1}), \varphi\right)$ of the segment $\eta\left(\left[\frac{1}{s}, 1\right]\right)$ in Σ_c such that $\eta\left(\frac{1}{s}\right)$ is an element of a convex neighbourhood of p in M. The construction is as follows.*

Consider $\mathcal{V} \subset \mathbb{R}^{n-2}$ to be a neighbourhood of 0 such that $\forall \alpha \in \mathcal{V} : (e_0 + \mathbf{e}(\alpha)) \in \tilde{\mathfrak{U}}$. Now let f be the map defined by

$$\begin{array}{rccl} f: & \mathcal{V} \subset \mathbb{R}^{n-2} & \to & \mathcal{C}_p(\mathcal{U}) \\ & \alpha & \mapsto & \exp_p\left(e_0 + \mathbf{e}(\alpha)\right). \end{array}$$

Then the map

$$\begin{array}{rccc} \varphi^{-1}: & (0-\epsilon, t_0+\epsilon) \times \mathcal{V} & \to & \Sigma_c \\ & (t, \alpha) & \mapsto & \Phi\left(t, f(\alpha)\right) \end{array}$$

is a diffeomorphism onto its image and $\left(\mathrm{im}(\varphi^{-1}), \varphi\right)$ is a coordinate neighbourhood of the geodesic segment $\exp_p\left([1, s]\tilde{u}\right) = \eta\left(\left[\frac{1}{s}, 1\right]\right)$.

2 $\tilde{u} = \kappa(e_1 + e_0)$ with $\kappa > 0$ can be obtained by a simple $SO(n-1)$ transformation of the spacelike component in the frame. Now a Lorentz transformation has to be performed. With $\theta := \log(\kappa)$, $\tilde{e}_1 = \cosh(\theta)e_1 + \sinh(\theta)e_0$ and $\tilde{e}_0 = \cosh(\theta)e_0 + \sinh(\theta)e_1$ are sufficient base vectors.

3 This choice is possible, since e_1 is orthogonal to e_i for all $i > 1$. Now consider e_1 as north pole of the sphere $\{x^1 e_i + \dots + x^{n-1} e_{n-1} \mid \|(x^1, \dots, x^{n-1})\| = 1\}$. The stereographic projection to the plane spanned by $\{e_2, \dots, e_{n-1}\}$ will have the desired property. Here we canonically identified the tangent space T_pM with $\mathbb{R}^n$.

Before going on with the proof, we will point out some important facts. First consider the case where t_0 is negative. Then we have to interchange the boundaries of the interval in the definition of φ^{-1}. The proof then essentially is the same. Next we observe that as result of the construction above the map φ has the following properties

$$\varphi^{-1}(0,0) = \Phi(0,x) = x = \exp_p(\tilde{u})$$
$$\varphi^{-1}(t_0,0) = \Phi(t_0,x) = q = \exp_p(s\tilde{u}).$$

Hence $\varphi^{-1}(t,0) = \Phi(t,x)$ with $t \in [0,t_0]$ is a pregeodesic from x to q and therefore the segment $\exp_p([1,s]\tilde{u})$ is a subset of $\operatorname{im}(\varphi^{-1})$.

Proof: We define $S := f(\mathcal{V})$. Then S is a smooth submanifold of Σ_c containing $x = f(0)$. Now let $\alpha_0 \in \mathcal{V}$. We will show the tangent space $T_{f(\alpha_0)}\Sigma_c$ to be a direct sum $T_{f(\alpha_0)}\Sigma_c = T_{f(\alpha_0)}S \oplus \langle \operatorname{grad}\sigma_{f(\alpha_0)} \rangle$. For that consider a curve $\eta(t) = f(\alpha_0 + t \cdot \alpha)$ in S with $\eta(0) = f(\alpha_0)$. Then $\dot{\eta}(0) = d\left(\exp_p\right)_{e_0+\mathbf{e}(\alpha_0)}(Y)$, where $Y = \partial_t|_{t=0}\mathbf{e}(\alpha_0 + t\alpha)$ by definition is a spacelike vector tangent to the null cone $\mathfrak{C}_pM$. In particular Y is non-null and therefore transversal to the null direction of the cone. Since $\alpha_0 + t\alpha \in \widetilde{\mathfrak{U}}$, $d\left(\exp_p\right)_{e_0+\mathbf{e}}$ is bijective and consequently $d\left(\exp_p\right)_{e_0+\mathbf{e}(\alpha_0)}(Y)$ is transversal to the null direction of Σ_c in $f(\alpha_0)$. As stated before the null direction on Σ_c is given by $\operatorname{grad}\sigma_{f(\alpha_0)}$. The function f itself is a diffeomorphism, since it is a composition of diffeomorphisms. Therefore we conclude that $T_{f(\alpha_0)}S = df_{\alpha_0}(\mathbb{R}^{n-2})$, which is transversal to $\operatorname{grad}\sigma_{f(\alpha_0)}$. This gives the claim.

We will now show $S \cap \Phi(t,S) = \emptyset$ for all $t \neq 0$. Let $y_1, y_2 \in S$ and assume $y_2 = \Phi(t,y_1)$ for some t, where without loss of generality $t \geq 0$. If $t < 0$ interchange y_1 and y_2. By definition of S we have $y_i \in S \subset \exp_p(\widetilde{\mathfrak{U}})$. Consequently with respect to the frame constructed above there are $\alpha_1, \alpha_2 \in \mathcal{V}$ such that $y_i = \exp_p(e_0 + \mathbf{e}(\alpha_i))$ for $i = 1,2$. Moreover, $\Phi(\cdot, y_2)$ is a null pregeodesic along radial geodesics of $\exp_p$ from y_1 to y_2. As both points are in the convex domain, there must be an $s \in \mathbb{R}$ such that $e_0 + \mathbf{e}(\alpha_2) = s\left(e_0 + \mathbf{e}(\alpha_1)\right)$. Due to the construction of $\mathbf{e}$, $\mathbf{e}(\alpha_i)$ and e_0 are orthogonal and we conclude $s = 1$. Therefore $\mathbf{e}(\alpha_1) = \mathbf{e}(\alpha_2)$ and since $\mathbf{e}$ is a diffeomorphism we get $\alpha_1 = \alpha_2$. Finally y_1 and y_2 coincide and we have $y_1 = \Phi(t,y_1)$. By Lemma 1.1.19 maximal integral curves of $\operatorname{grad}\sigma$ originating in or heading for a vertex are complete and non-self-intersecting such that we get the desired result, namely $t = 0$.

Finally S, f and Φ comply with the requirements of Lemma 5.1.19 such that φ^{-1} is a diffeomorphism and hence φ is a chart as claimed. ∎

Lemma 5.1.31. *Consider the chart $\varphi : \mathcal{D} \subset \Sigma \to \mathbb{R}^{n-1}$ defined above. Let $X \in \mathfrak{C}_pM$, $s \in \mathbb{R}$ such that X and sX are in the domain of $\exp_p$ and $\exp_p(X), \exp_p(sX) \in \mathcal{D}$. Denote $\varphi = (\varphi_0, \ldots, \varphi_{n-2})$, then we find the following properties*

$$\forall i \neq 0: \quad \varphi_i\left(\exp_p(X)\right) = \quad \varphi_i\left(\exp_p(sX)\right) \tag{5.13}$$

and with $\tilde{u} \in \tilde{\mathcal{U}}$ as defined above (in particular $\exp_p(\tilde{u}) = x = \varphi^{-1}(0)$) we get

$$\operatorname{rank}\left(d\left(\exp_p\right)_{\tilde{u}} : T_{\tilde{u}}\left(\mathfrak{C}_pM\right) \to T_{\varphi^{-1}(0)}\Sigma_c\right) = n-1. \tag{5.14}$$

Proof: The second statement follows from the definition of the coordinates. The point $\tilde{u}$ is in the neighbourhood of $0 \in T_pM$ where the exponential map is a diffeomorphism such that the restriction to a submanifold is a diffeomorphism too.

For the first statement consider X and s as required. Let

$$\exp_p(X) = \varphi^{-1}(\kappa_0, \alpha) = \Phi(\kappa_0, \exp_p(e_0 + \mathbf{e}(\alpha))) \tag{5.15}$$

be the unique coordinate representation of $\exp_p(X)$ and define the null geodesic $\eta(t) = \exp_p(tX)$ for $t \in [1,s]$. Then due to the assumptions we have $\eta(t) \in \Sigma_c$ for all $t \in [1,s]$. Therefore

$\dot{\eta}(t) \propto \operatorname{grad} \sigma_{\eta(t)} \neq 0$ such that there is an integral curve of $\operatorname{grad} \sigma$ connecting $\exp_p(X)$ and $\exp_p(sX)$. Hence

$$\begin{aligned}
\exp_p(sX) &= \Phi(\kappa_1, \exp_p(X)) \\
&\overset{(5.15)}{=} \Phi\left(\kappa_1, \Phi(\kappa_0, \exp_p(e_0 + \mathbf{e}(\alpha)))\right) \\
&= \Phi\left(\kappa_0 + \kappa_1, \exp_p(e_0 + \mathbf{e}(\alpha))\right) \\
&= \varphi^{-1}(\kappa_0 + \kappa_1, \alpha).
\end{aligned}$$

■

In general the exponential map is not a diffeomorphism on its domain and it is convenient to consider only the subset of the tangent space where it is a diffeomorphism. Assume p to be a vertex of Σ, then we get the following lemma on the intersection of the latter domain and the tangent null cone $\mathfrak{C}_p M$. domain.

Lemma 5.1.32. *Let (M, g, σ) be an almost Einstein structure and Σ its singularity set. Consider $p \in \Sigma_d$ and let $\gamma : (0, 1] \to \Sigma_c$ be the null geodesic defined by $\gamma(t) = \exp_p(tu)$ with $u \in \mathfrak{C}_p M$ and such that there is no further vertex in $\gamma((0, 1])$. Then there is a set $\Omega \subset \mathfrak{C}_p M \cup \{0\}$ such that*

(i) Ω is star shaped and $\Omega \setminus \{0\}$ is open in $\mathfrak{C}_p M$

(ii) $\forall t \in [0, 1] : \quad \gamma(t) \in \exp_p(\Omega)$ and

(iii) $\exp_p : \Omega \setminus \{0\} \to \Sigma_c$ is a diffeomorphism onto its image.

Proof: Basically we will use the same construction methods as above. The first step will be to construct the neighbourhood Ω of the the preimage of the geodesic. Then we will show $\exp_p$ to be a diffeomorphism on it.

First we point out that the null geodesic γ can be extended to values $t \in (0 - \epsilon, 1 + \epsilon)$. The extension to values $t < 0$ is possible, since $\exp_p$ is a local diffeomorphism in p. The extension to values $t > 1$ is possible, since for $\gamma(1) \in \Sigma_c$ the gradient $\operatorname{grad} \sigma_{\gamma(1)}$ does not vanish. Therefore the integral curve in $\gamma(1)$ can locally be extended in both directions. Since it is a null pregeodesic tangent to γ, it can be reparametrised to a null geodesic tangent to γ. By an affine transformation of the parameter this gives the desired extension of γ beyond 1.

We will now use the notation introduced in the preliminaries of Lemma 5.1.30. So let $\mathfrak{U}$ be a convex neighbourhood of the origin in $T_p M$, such that p is the only critical point of $\operatorname{grad} \sigma$ in $\exp(\mathfrak{U})$. Then $\operatorname{grad} \sigma_{\exp_p(X)} = 0$ for $X \in \mathfrak{U}$ if and only if $X = 0$. The end point of γ will again be denoted $q = \exp_p(u)$ and is an element of Σ_c. We define

$$\Omega_1 := (\mathfrak{C}_p M \cap \mathfrak{U}) \cup \{0\}$$

to be the tangent null cone including its vertex. If we restrict the exponential map to Ω_1 without the vertex, i.e.

$$\exp_p : \Omega_1 \setminus \{0\} \to \Sigma_c,$$

this is a restriction to an $(n-1)$-dimensional submanifold of $T_p M$ and hence a diffeomorphism onto its image. By convexity of $\mathfrak{U}$ we get that Ω_1 is star shaped with respect to the origin.

We recall the preceding definition, i.e. there is a $\tilde{u} \in \mathfrak{U}$ with $x = \exp_p(\tilde{u}) \in \Sigma_c$ such that $u = s\tilde{u}$. The gradient $\operatorname{grad} \sigma$ does not vanish along γ and is tangent to it. By Lemma 5.1.6(iiv) γ can be reparametrised to an integral curve of $\operatorname{grad} \sigma$ and hence there is a $t_0 \in \mathbb{R}$ such that $\gamma(1) = \exp_p(u) = \Phi(t_0, x)$. We consider the case where p is a repeller and therefore $t_0 > 0$. For the other case we have negative t_0 and the roles of x and q have to be swapped. By Lemma 5.1.30 there is a coordinate map

$$\varphi : \widetilde{W} \to (-\epsilon, t_0 + \epsilon) \times \mathcal{V} \subset \mathbb{R}^{n-1}$$

with $\widetilde{W} := \varphi^{-1}((0-\epsilon, t_0+\epsilon)\times\mathcal{V})$ such that the segment of γ containing x and $q = \gamma(1)$ is contained in $\widetilde{W}$. By construction the preimage $\varphi^{-1}(t,\alpha)$ is in $\tilde{\mathcal{U}}$ for $t \in (-\epsilon, 0]$ such that in particular $\varphi^{-1}(t,\alpha) \in \exp_p\left(\tilde{\mathfrak{U}}\right) \subset \exp_p(\Omega_1)$ for all $t \leq 0$. Since $\varphi(\cdot,\alpha) = \Phi(\cdot, x)$ are null pregeodesics tangent to radial null geodesics starting at p, the exponential map is defined onto points in $\widetilde{W}$. That is why

$$\Omega_2 := \exp_p^{-1}\left(\widetilde{W}\right)$$

is a well defined quantity. Again we remark that by construction of φ^{-1} the vector field $\operatorname{grad}\sigma$ does not vanish along $\widetilde{W}$. We define

$$\Omega := \Omega_1 \cup \Omega_2.$$

The exponential map is defined for all $X \in \Omega$ as seen above and we will show that Ω has the desired properties.

We find

$$\operatorname{grad}\sigma_{\exp_p(X)} = 0 \quad\Leftrightarrow\quad X = 0, \tag{5.16}$$

since this holds for $X \in \Omega_1$ and $X \in \Omega_2$. Also $\Omega_1 \setminus \{0\}$ and Ω_2 are by construction open in $\mathfrak{C}_p M$ and so is their union.

Now we show Ω to be star shaped with respect to 0. The claim is clear for $X \in \Omega_1$. Now consider $X \in \Omega_2$. Then $\exp_p(X) = \varphi^{-1}(t,\alpha)$ and hence

$$\exp_p(X) = \Phi(t, f(\alpha)) = \Phi(t, \exp_p(Y))$$

for some $Y \in \Omega_1$, since $\operatorname{im}(f) \subset \Omega_1$. This implies that there is a null pregeodesic in $\widetilde{W}$ from $\exp_p(Y)$ to $\exp_p(X)$. Hence $X = \mu Y$ for some $\mu > 1$ and therefore $[1,\mu]Y \subset \Omega_2$. Combining it with the last observation, this leads to $[0,\mu]Y = [0,1]X \subset \Omega$.

Second we show $\exp_p : \Omega \to \Sigma$ to be bijective onto its image. Assume $\exp_p(X_1) = \exp_p(X_2)$ for $X_i \in \Omega$ and define the geodesics $\gamma_i(t) := \exp_p(tX_i)$. Since Ω is star shaped we have $tX_i \in \Omega$ for all $0 \leq t \leq 1$. By construction $\operatorname{grad}\sigma$ is non-vanishing along the geodesics, except in $p = \gamma_i(0)$. Therefore there are non-vanishing functions $f_i : (0,1] \to \mathbb{R}^{\pm}$ along the geodesics such that $f_i(t)\dot{\gamma}_i(t) = \operatorname{grad}\sigma_{\gamma_i(t)}$. By Lemma 5.1.8 p is an attractor or an repeller of $\operatorname{grad}\sigma$. Since $\operatorname{grad}\sigma$ is tangent to null geodesic radiating from p this means that f_1 and f_2 must have the same sign such that in particular $\mu := \frac{f_1(1)}{f_2(1)}$ is strictly positive. Without loss of generality assume $\mu \in (0,1]$ and interchange the roles of X_1 and X_2 else. At $\gamma_1(1) = \gamma_2(1)$ we therefore have

$$\dot{\gamma}_2(1) = \frac{1}{f_2(1)}\operatorname{grad}\sigma_{\gamma_1(1)} = \mu\dot{\gamma}_1(1).$$

Now consider the geodesic $\eta(t) := \mu_1(\mu(t-1)+1)$. Then it holds $\eta(1) = \gamma_1(1) = \gamma_2(1)$ and $\dot{\eta}(1) = \mu\dot{\gamma}_1(1) = \dot{\gamma}_2(1)$. By uniqueness of geodesics we then have $\eta(t) = \gamma_2(t)$ for all $t \in [0,1]$. In particular $p = \mu_2(0) = \eta(0) = \mu_1(1-\mu)$ with $1-\mu < 1$. Since we assumed $X_1 \in \Omega$, by condition (5.16) this is equivalent to $\mu = 1$ and hence $\mu_1(t) = \mu_2(t)$ for all t. We conclude $X_1 = X_2$.

The third part of the proof will be to show that $d\left[\exp_p\right]_u$ is an isomorphism for an arbitrary $u \in \Omega \setminus \{0\}$. It suffices to show $\ker\left(d\left[\exp_p\right]_u\right) = \{0\}$. Since $\Omega\setminus\{0\}$ is open in $\mathfrak{C}_p M$, there is an $\epsilon > 0$ such that $\gamma : t \mapsto \exp_p(tu)$ is well defined for all $t \in (-\epsilon, 1+\epsilon)$. Then according to Lemma 5.1.30 there is an open set $U_q \subset \Sigma_c$ and coordinates $\varphi : U_q \to \mathbb{R}^{n-1}$ such that the properties of Lemma 5.1.31 are fulfilled. Following the preceding construction there is a orthonormal frame $\{e_i\}$ of $T_p M$ and $\tilde{u} \in \mathfrak{C}_p M$ such that

$$\begin{aligned} u &= s\tilde{u} & \qquad \exp_p(\tilde{u}) &= \varphi(0) \\ \tilde{u} &= e_0 + e_1 & \exp_p(u) &= \varphi(t_0, 0) \end{aligned}$$

and $d\left[\exp_p\right]_{\tilde{u}}$ is an isomorphism.

We will now derive $d\left[\exp_p\right]_u$ in the coordinates φ. Consider $\alpha := (\alpha^2, \ldots, \alpha^{n-1}) \in \mathbb{R}^{n-2}$ and $\alpha^n \in \mathbb{R}$ both sufficiently small. Then by construction of $\mathbf{e}$ (see Equation (5.12)) the curves

$$\begin{aligned}\eta^n(t) &:= (1+\alpha^n t)(e_0+e_1)\\ \eta^s(t) &:= e_0 + \mathbf{e}(t\alpha)\end{aligned}$$

are curves in the null cone $\mathfrak{C}_p M$ with $\eta^s(0) = \eta^n(0) = \tilde{u}$. Moreover, $\dot{\eta}^n(0) = \alpha^n \tilde{u}$ and $\dot{\eta}^s(0) = \alpha^2 e_2 + \cdots + \alpha^{n-1} e_{n-1}$ are tangent to the null cone at $\tilde{u}$. Consequently the scaled curves $s\eta^n$ and $s\eta^s$ are also tangent to the null cone at $u = s\tilde{u}$. Curves of type η^s and η^n generate all spacelike or all null tangent vectors in $\tilde{u}$, while $s\cdot\eta^s$ and $s\cdot\eta^n$ do the same for tangent vectors in u.

Now consider the following composition

$$T_u\left(\mathfrak{C}_p M\right) \xrightarrow{d\left[\exp_p\right]_u} T_{\exp_p(u)}\Sigma_c \xrightarrow{d\varphi_{\exp_p(u)}} \mathbb{R}^{n-1} \simeq T_{\varphi(q)}\mathbb{R}^{n-1}$$

In the following we will identify the tangent spaces $T_u\left(\mathfrak{C}_p M\right)$ and $T_{\tilde{u}}\left(\mathfrak{C}_p M\right)$ with each other. For null tangent vectors $\dot{\eta}^n$ we find

$$\begin{aligned}\left(d\varphi_{\exp_p(u)} \circ d\left[\exp_p\right]_u\right)(\dot{\eta}^n(0)) &= \frac{1}{s}\left(d\varphi_{\exp_p(u)} \circ d\left[\exp_p\right]_u\right)(s\dot{\eta}^n(0))\\ &= \frac{\alpha^n}{s} d\varphi_{\exp_p(u)}\left(d\left[\exp_p\right]_u(u)\right)\\ &= \frac{\alpha^n}{s} d\varphi_{\exp_p(u)}\left(\dot{\gamma}(1)\right).\end{aligned}$$

Since $d\varphi_{\exp_p(u)}$ is an isomorphism, the last line does not vanish. Furthermore we have

$$\begin{aligned}d\varphi_{\exp_p(u)}\left(d\left[\exp_p\right]_u(u)\right) &= \left.\frac{d}{dt}\right|_{t=0} \varphi\left(\exp_p(u+tu)\right)\\ &= \left.\frac{d}{dt}\right|_{t=0} (g(t), 0, \ldots, 0)\end{aligned}$$

for some smooth map g, since the coordinates along the geodesic γ have only a non-vanishing component at the first position. From the previous calculation we conclude that $\dot{f}(0) \neq 0$. Moreover, we point out that this calculation is in principle valid for arbitrary base points as $d\left[\exp_p\right]_u(u)$ is tangent to the geodesic specified by $\exp_p(tu)$.

For spacelike tangent vectors $\dot{\eta}^s$ we then find for the i-th component ($i \neq 0$) of its image under this composition

$$\begin{aligned}\left[\left(d\varphi_{\exp_p(u)} \circ d\left[\exp_p\right]_u\right)(\dot{\eta}^s(0))\right]_i &= \frac{1}{s}\left[\left(d\varphi_{\exp_p(u)} \circ d\left[\exp_p\right]_u\right)(s\cdot\dot{\eta}^s(0))\right]_i\\ &= \frac{1}{s}\left.\frac{d}{dt}\right|_{t=0} \varphi_i\left[\exp_p(s\cdot\eta^s(t))\right]\\ &\overset{(5.13)}{=} \frac{1}{s}\left.\frac{d}{dt}\right|_{t=0} \varphi_i\left[\exp_p(\eta^s(t))\right]\\ &\overset{i\neq 0}{=} \frac{1}{s}\left[\left(d\varphi_{\exp_p(\tilde{u})} \circ d\left[\exp_p\right]_{\tilde{u}}\right)(\dot{\eta}^s(0))\right]_i\end{aligned}$$

where the last line is non-vanishing for some $i \in \{1, \ldots, n-2\}$. Assume the contrary and let the last line vanish for all $i \in \{1, \ldots, n-2\}$ then the image of a spacelike and a null vector could be added to give a Zero vector, which would contradict $d\varphi_{\exp_p(\tilde{u})} \circ d\left[\exp_p\right]_{\tilde{u}}$ to be an isomorphism.

We conclude that $\ker\left(d\varphi_{\exp_p(u)} \circ d\left[\exp_p\right]_u\right) = \{0\}$. Now $d\varphi_{\exp_p(u)}$ is an isomorphism and hence $d\left[\exp_p\right]_u$ is an isomorphism too. ∎

The last lemmata can be summarised to classify the maximal domain where the restricted exponential map is a diffeomorphism.

Corollary 5.1.33. *Let $p \in \Sigma_d$ and $\tilde{\mathfrak{U}}$ open in $\mathfrak{C}_p M$ such that $\tilde{\mathfrak{U}} \cup \{0\}$ is star shaped with respect to the origin and $\exp_p\big|_{\tilde{\mathfrak{U}}} : \tilde{\mathfrak{U}} \to \Sigma_c$ is a diffeomorphism. Let $\mathcal{D}^\Phi \subset \mathbb{R} \times M$ be the maximal domain of the flow Φ of $\operatorname{grad} \sigma$. Then $\tilde{\mathfrak{U}}_{max} := \exp_p^{-1}\left(\Phi\left(\mathbb{R} \times \tilde{\mathfrak{U}} \cap \mathcal{D}^\Phi\right)\right)$ is a well defined extension of $\tilde{\mathfrak{U}}$ and $\exp_p\big|_{\tilde{\mathfrak{U}}} : \tilde{\mathfrak{U}}_{max} \to \Sigma_c$ is a diffeomorphism.*

The maximal subset cannot be bigger since every radial null geodesic that leaves a connected component of Σ_c must leave it at a vertex by Lemma 5.1.7. More precisely as long as the null geodesic has values in Σ_c it can be reparametrised to an integral curve of $\operatorname{grad} \sigma$. Since there is a point where the null geodesic is not in Σ_c, the reparametrised integral curve cannot be extended beyond that point and therefore must be maximal. Now Lemma 5.1.7 can be applied.

Remark. We assert that any subset of Σ that is a spacelike $(n-2)$-dimensional submanifold of M must not contain any of the vertices in Σ_d. This obviously has the same reason that prevent Σ from being a submanifold of M at those points. Hence a spacelike submanifold of Σ implicitly is a submanifold of Σ_c.

Proposition 5.1.34. *Let N be an $(n-2)$-dimensional spacelike submanifold of Σ_c and $\gamma : [0, t_0] \to \Sigma$ a null geodesic with $\gamma(0) := q \in N$, $\gamma(t_0) =: p \in \Sigma_d$ and $\gamma([0, t_0)) \subset \Sigma_c$. In particular there is no other vertex between $\gamma(0)$ and $\gamma(t_0)$. Then $\gamma(t_0)$ is a focal point of N with respect to γ.*

Proof: First for $\epsilon > 0$ being sufficiently small, γ can be extended to values $t \in (-\epsilon, t_0 + \epsilon)$. Moreover, since $\gamma(t_0) = p$, there is a $u \in \mathfrak{C}_p M$ with $\gamma(t) = \exp((t - t_0)u)$. We are now in the setting of Lemma 5.1.32 and there is an $\Omega \subset \mathfrak{C}_p M$ containing $\gamma([0, t_0])$ such that the exponential map is a diffeomorphism on $\Omega \setminus \{0\}$.

We will now construct a geodesic variation $\delta\gamma$ normal to N with vanishing Jacobi field at $\gamma(t_0) = p$. Consider $\delta > 0$ sufficiently small and a curve $\alpha : (-\delta, \delta) \to N$ with $\alpha(0) = \gamma(0)$, $\alpha(s) \in \Omega$ for all $s \in (-\delta, \delta)$ and $\alpha'(0) \neq 0$. We define

$$\begin{array}{rccl} X : & (-\delta, \delta) & \to & \mathfrak{C}_p M \\ & s & \mapsto & \exp_p^{-1}(\alpha(s)). \end{array}$$

Due to Lemma 5.1.32 this map is well defined and depends smoothly on the parameter s. Its covariant derivative with respect to α is

$$X'(0) = \frac{\nabla}{ds} X(0) = d\left[\exp_p^{-1}\right]_{\alpha(0)} \left(\alpha'(0)\right)$$

and it is non-vanishing as the exponential map is a diffeomorphism and $\alpha'(0) \neq 0$. Up to rescaling of δ and ϵ, the geodesic variation of γ

$$\begin{array}{rccl} \delta\gamma : & (-\epsilon, t_0 + \epsilon) \times (-\delta, \delta) & \to & \Sigma \\ & (t, s) & \mapsto & \exp_p\left(\frac{t_0 - t}{t_0} X(s)\right). \end{array}$$

is well defined. By definition we have $\delta\gamma(0, s) = \alpha(s)$ and $\delta\gamma(t_0, \cdot) \equiv p$. In addition $\delta\gamma(\cdot, s)$ are null geodesics in Σ for fixed s and therefore normal to N. We now define its variation vector field

$$\begin{array}{rccl} J : & [0, t_0] & \to & TM \\ & t & \mapsto & \partial_s \delta\gamma(t, 0). \end{array}$$

More explicitly this reads

$$\begin{aligned} J(t) = {} & \partial_s|_{s=0} \exp_p\left(\frac{t_0 - t}{t_0} X(s)\right) \\ = {} & d\left[\exp_p\right]_{\frac{t_0 - t}{t_0} X(0)} \left(\frac{t_0 - t}{t_0} X'(0)\right). \end{aligned}$$

We remark that $X'(0) \neq 0$ is tangent to $\mathfrak{C}_p M$ and $\frac{t_0 - t}{t_0} X(0) \in \Omega$. Therefore $d\exp_p$ is an isomorphism at least on the tangent space of the null cone and we get $J(t) = 0$ if and only if $t = t_0$. Summarising the facts we conclude that J is a N-Jacobi field on γ and $\gamma(t_0) = p$ a focal point as claimed. ■

Now summarising Propositions 5.1.29 and 5.1.34 gives the following theorem.

Theorem 5.1.35. *Let (M, g, σ) be an almost Einstein structure with $S[g, \sigma] = 0$. Let N be an $(n-2)$-dimensional spacelike submanifold of Σ_c and $\gamma : [0, t_0] \to \Sigma$ a null geodesic with $\gamma(0) := q \in N$ and $\gamma([0, t_0)) \subset \Sigma_c$. In particular there is no vertex between $\gamma(0)$ and $\gamma(t_0)$. Then $\gamma(t_0)$ is a focal point of N with respect to γ if and only if it is a vertex of Σ. In other words*

$$\gamma(t_0) \text{ is a focal point of } N \iff \gamma(t_0) \in \Sigma_d.$$

5.2 ASYMPTOTIC STRUCTURE OF Σ

5.2.1. Asymptotic Behaviour Near Σ

First we have to define what it means to hit Σ.

Definition 5.2.1. Let (M, g, σ) be an almost Einstein structure and $\tilde{g} = \sigma^{-2} g$ the Einstein metric on $\tilde{M} = M \setminus \Sigma$. A geodesic $\tilde{\gamma} : (\tilde{a}, \tilde{b}) \to \tilde{M}$ will be said to hit Σ in $\tilde{\gamma}(\tilde{a})$ or in $\tilde{\gamma}(\tilde{b})$ if there is a smooth curve $\gamma : (a, b) \to M$ and a reparametrisation $h : (\tilde{a}, \tilde{b}) \to (a, b)$ with the following properties

(i) $\tilde{\gamma} = \gamma \circ h$

(ii) $a < h(\tilde{a})$ or $h(\tilde{b}) < b$. If $\tilde{a}$ or $\tilde{b}$ are not real numbers, the inequality is understood as limit. Moreover $\gamma \circ h(\tilde{a}) \in \Sigma$ or $\gamma \circ h(\tilde{b}) \in \Sigma$

(iii) $\dot{\gamma}(t) \neq 0$ for all $t \in (a, b)$.

The curve γ then is a pregeodesic with respect to $\tilde{\nabla}$. We introduce the notation

$$\tilde{I} := (\tilde{a}, \tilde{b}) \qquad\qquad I' := h(\tilde{I})$$

for the intervals under consideration.

Lemma 5.2.2. *Let $\tilde{\gamma} : I \to \tilde{M}$ be a null geodesic with respect to $\tilde{\nabla}$ that hits Σ at $p := \tilde{\gamma}(\tilde{b})$. Then $\tilde{\gamma}(\tilde{b})$ is an element of Σ_c*

Proof: By Lemma 1.4.5 $\tilde{\gamma}$ can be reparametrised to a geodesic $\gamma : I' \to M$ with $\gamma \circ h(t) \to p \in \Sigma$ for $t \to \tilde{b}$. Hence by Lemma 1.2.4 γ can be extended to a geodesic in p and therefore is a radial null geodesic in p. Hence let $\mathcal{U} \subset M$ be a normal neighbourhood of p then the image $\mathrm{im}(\gamma) \cap \mathcal{U}$ of γ inside $\mathcal{U}$ is a subset of the geodesic null cone $\mathcal{C}_p(\mathcal{U})$. Consider $p \in \Sigma_d$. Therefore by Proposition 5.1.12 we can shrink $\mathcal{U}$ such that $\mathcal{C}_p(\mathcal{U}) = \Sigma \cap \mathcal{U}$. Then at that neighbourhood γ is a curve without any point in $\tilde{M} = M \setminus \Sigma$ and so cannot be a reparametrisation of $\tilde{\gamma}$ within that neighbourhood of p. Consequently we find $p \in \Sigma_c$. ■

Corollary 5.2.3. *Any $\tilde{g}$-geodesic that hits a point in Σ_d is spacelike or timelike.*

Proof: Assume $\tilde{\gamma} : I \to \tilde{M}$ to be a null geodesic with limit $\tilde{\gamma}(\tilde{b}) \in \Sigma_d$. Then the same argument used in the previous lemma leads to a contradiction, hence such null geodesics do not exist. ■

Lemma 5.2.4. *Any $\tilde{g}$-null geodesic that hits Σ_c has a tangent vector which is transversal to Σ_c at that point.*

Proof: Let $\tilde{\gamma}$ be a $\tilde{g}$-null geodesic. Then by Lemma 1.4.5 it has a reparametrisation γ to a g-geodesic. Assume $\gamma(t_0) \in \Sigma_c$ is the point where $\tilde{\gamma}$ hits Σ_c and $\dot{\gamma}(t_0) \in T_{\gamma(t_0)}\Sigma$. Hence the null vector $\dot{\gamma}(t_0)$ is collinear to $\mathrm{grad}_{\gamma(t_0)}$. The corresponding integral curve of $\mathrm{grad}\,\sigma$ is at least locally completely within Σ_c by Lemma 5.1.6(iii) and can be reparametrised to a g-null geodesic by statement (iv) of that lemma. By uniqueness of geodesics this reparametrisation coincides with γ up to rescaling with a constant factor. Therefore γ at least locally is completely within Σ and cannot be a reparametrisation of $\tilde{\gamma}$. ∎

Lemma 5.2.5. *Any space- or timelike $\tilde{g}$-geodesic $\tilde{\gamma}$ that hits Σ_c, hits it with null tangent vector.*

Proof: Let be $\gamma : I' \to M$ the reparametrisation of $\tilde{\gamma}$with $\gamma(t_0) \in \Sigma_c$ being the point, where $\tilde{\gamma}$ hits Σ. Then γ is a $\tilde{g}$-pregeodesic, since h provides the reparametrisation to a geodesic. Hence there is a function $c : I' \to \mathbb{R}$ such that $\tilde{\nabla}_{\dot{\gamma}}\dot{\gamma} = c\dot{\gamma}$. The reparametrisation exists only for values of t with $\gamma(t) \in \tilde{M}$. Using the conformal transformation behaviour of the Levi-Civita connection in Equation (1.101) we get

$$\begin{aligned} c(t)\dot{\gamma} &= \tilde{\nabla}_{\dot{\gamma}}\dot{\gamma} \\ &= \nabla_{\dot{\gamma}}\dot{\gamma} - 2\sigma^{-1} d\sigma(\dot{\gamma})\dot{\gamma} + \sigma^{-1} g(\dot{\gamma},\dot{\gamma})\,\mathrm{grad}\,\sigma. \end{aligned}$$

Apart from Σ this is equivalent to

$$(2d\sigma(\dot{\gamma}) + c(t)\sigma)\,\dot{\gamma} = g(\dot{\gamma},\dot{\gamma})\,\mathrm{grad}\,\sigma + \sigma\nabla_{\dot{\gamma}}\dot{\gamma} \tag{5.17}$$

The right-hand side can be smoothly extended to Σ and equals $g(\dot{\gamma},\dot{\gamma})(t_0)\,\mathrm{grad}\,\sigma_{\gamma(t_0)}$ at $\gamma(t_0)$. Hence the left-hand side must have the same extension to Σ. Assume $\dot{\gamma}(t_0)$ to be non-null, then the right-hand side is a non-vanishing null vector. The left-hand side is tangent to γ for all $t \in I'$, with $\gamma(t) \notin \Sigma$. Hence the limit on the left-hand side must be tangent to γ in t_0 and therefore is a non-null vector or vanishes. Both contradict the equality of left- and right-hand side in the equation for all neighbourhoods of $\gamma(t_0)$. Thus $\dot{\gamma}(t_0)$ is a null vector, in particular $g(\dot{\gamma}(t_0), \dot{\gamma}(t_0)) = 0$. ∎

Corollary 5.2.6. *Moreover by requirement of $\dot{\gamma}$ not to vanish we conclude*

$$2d\sigma(\dot{\gamma}) + c\sigma \to 0$$

for $t \to t_0$ or $\gamma(t) \to \gamma(t_0)$ and hence the map c is divergent in that limit.

Lemma 5.2.7. *The Weyl tensor vanishes at all vertices $p \in \Sigma_d$.*

Proof: By using Equation (1.125) we have $\mathrm{grad}\,\sigma \lrcorner \mathrm{W} =|_{\Sigma} 0$. By Proposition 5.1.12, there is a normal neighbourhood U of p such that $\Sigma \cap \mathcal{U}$ coincides with the geodesic null cone $\mathcal{C}_p(U)$ in p. Therefore

$$\mathrm{W}(\mathrm{grad}\,\sigma, \dots) =|_{\mathcal{C}_p(U)} 0.$$

If U is chosen sufficiently small the gradient vector field $\mathrm{grad}\,\sigma$ satisfies

$$\begin{aligned} \mathrm{grad}\,\sigma|_{U\setminus\{p\}} \neq 0 \qquad & \|\mathrm{grad}\,\sigma\|^2\Big|_{\mathcal{C}_p(U)} = 0 \\ \mathrm{grad}\,\sigma_x \in T_x\mathcal{C}_p(U) \quad & \text{for } x \in \mathcal{C}_p(U). \end{aligned}$$

Hence the Weyl tensor and $\mathrm{grad}\,\sigma$ fulfil the requirements of Proposition 1.2.6 such that we get $\mathrm{W}_p = 0$. ∎

Finally we can summarise the previous section.

Proposition 5.2.8. *The asymptotic behaviour in a neighbourhood of Σ is as follows. Let (M, g, σ) be a almost Einstein structure and $(M \setminus \Sigma, \tilde{g} = \sigma^{-2} g)$ the rescaled Einstein manifold, then*

(i) *for any $\tilde{g}$-null geodesic that hits Σ in a point p, that p is in Σ_c.*

(ii) *Any maximal g-null geodesic that is tangent to* Σ_c *in one point* $p \in \Sigma_c$ *is a null geodesic on* Σ *on its domain.*

(iii) *Any* $\tilde{g}$*-geodesic that hits* Σ *in* Σ_d *is a space- or timelike geodesic.*

(iv) *Any* $\tilde{g}$*-geodesic that hits* Σ_c *hits it with null tangent vector.*

(v) *The Weyl tensor vanishes at the vertices of* Σ.

Proof: All claims, except for the second one, have already been proven. For the second claim assume $\gamma : I = (\alpha, \beta) \to M$ to be a maximal null geodesic with $\gamma(t_0) \in \Sigma_c$ and $\dot{\gamma}(t_0) \in T_{\gamma(t_0)}\Sigma_c$. Fist we will point out that at least locally γ maps to Σ. Next this property will be shown to be a global one.

The vector $\dot{\gamma}(t_0)$ is a tangent null vector of Σ_c and hence proportional to $\operatorname{grad} \sigma_{\gamma(t_0)}$. Hence by Lemma 5.1.6(iii) the maximal integral curve of $\operatorname{grad} \sigma$, with origin at $\gamma(t_0)$, is completely within Σ_c. By Lemma 5.1.6(iv) it can be reparametrised to a geodesic η, which then too is completely within Σ_c. Its domain is an interval and will be denoted I. Without loss of generality the interval is chosen such that $\gamma(t_0) = \eta(t_0)$. By uniqueness the two geodesics must coincide on $\tilde{I} = (\tilde{\alpha}, \tilde{\beta}) \subset I$. Hence γ at least locally maps to Σ.

Next we show that a null geodesic on Σ may not leave it at its vertices. Assume $\lim_{t \to \tilde{\beta}} \gamma(t)$ to exist, then since η is the reparametrisation of a maximal $\operatorname{grad} \sigma$-integral curve the limit is a vertex $x \in \Sigma_d$ by Lemma 5.1.7. Hence the null geodesic γ coincides with the exponential map $\gamma(t) = \exp_x((t + t')X)$ for some null vector $X \in T_x M$ and $t' \in \mathbb{R}$. By Proposition 5.1.12 Σ locally coincides with the geodesic null cone in x. Hence γ has an extension to values greater than $\tilde{\beta}$, which locally is in the null cone and hence in Σ. Now one may start an inductive process. Starting at $\gamma(t_0)$, we can extend the interval $\tilde{I}$ from one vertex to the next one. If we assume extensions of the initial interval $\tilde{I}$ to bigger values of t, we may denote the first value of t where $\gamma(t) \in \Sigma_d$ with t_1, the second with t_2 and so on. If there are just finitely many such values, then the last part of the maximal interval I must be in Σ_c. The sequence of t_i is monotonic increasing by definition. Now assume $\{t_i\}$ to be an infinite sequence. If it is not bounded, the maximal geodesic is complete in positive direction and completely within Σ. If it is bounded, there is a limit and we have to show that

$$\lim_{i \to \infty} t_i = \beta.$$

Assume $\lim_{i \to \infty} t_i =: T < \beta$. Then we have that γ is a smooth map in a neighbourhood of T. Since $\gamma(t_i) \in \Sigma_d$ for all i, we consequently have the following limits

$$\lim_{i \to \infty} \sigma \circ \gamma(t_i) = 0 \qquad\qquad \lim_{i \to \infty} \operatorname{grad} \sigma_{\gamma(t_i)} = 0.$$

Hence $\gamma(T) \in \Sigma_d$. By properties 5.1.1, $\gamma(T)$ is an isolated point with a neighbourhood, not containing any other critical points of σ. Hence, there is a neighbourhood $(T - \epsilon, T + \epsilon)$ such that $\gamma(t) \notin \Sigma_d$ for all t in that neighbourhood. Hence T can not be limit of the t_i. This contradicts the definition of T and hence $T = \beta$. ■

5.3 SPECIAL COORDINATES

Now we will construct special coordinates for the neighbourhood of vertices in Σ. As already seen in Proposition 5.1.1, the Morse lemma provides coordinates such that Σ locally turns out to be a quadric. These coordinates still have a remaining freedom of choice. We will use it to induce an additional property, namely that null curves on Σ are mapped to straight lines in $\mathbb{R}^n$. This is achieved by a deformation of the Morse coordinates along the cone and showing that the method can be extended to a neighbourhood of the cone in a suitable way. The first part of this section introduces the method of deformation for points on a sphere. It can then naturally be extended to work on cylinders with boundary. Next we will introduce a way to make the deformation method work on double cones. We will then discover that this fortunately implies a method that even works on a neighbourhood of the cone, eventually by losing smoothness of the coordinates at the vertex.

Let $A \in \mathbb{R}^{m^2}$ be a matrix and v a vector in $\mathbb{R}^m$. Then throughout the section we will denote the canonic vector norm with $\|v\|$ and use the notation $\|A\| := \max\limits_{x \neq 0} \frac{\|A \cdot x\|}{\|x\|}$ for the induced matrix norm.

Lemma 5.3.1. *Let $U \subset \mathbb{R}^n$ be a neighbourhood of 0 such that the spheres*

$$S^{n-2}_{c_1,c_2} := \left\{ x \in \mathbb{R}^n \mid x^0 = c_1, \left(x^1\right)^2 + \cdots + \left(x^{n-1}\right)^2 = c_2 \right\} \simeq S^{n-2}$$

with constants c_1 and c_2 are completely either inside or outside U, i.e. either $S^{n-2}_{c_1,c_2} \cap U = S^{n-2}_{c_1,c_2}$ or $S^{n-2}_{c_1,c_2} \cap U = \emptyset$. Let $g : U \to \{0\} \times \underline{\mathfrak{so}}(n-1) \subset \underline{\mathfrak{so}}(n)$ be a C^0 map which is C^m on $U \setminus \{0\}$ and has the properties

$$\begin{array}{llcl} (i) & g(0) & = & 0 \\ (ii) & \|d\exp_{g(x)}\| & \leq & 2 \\ (iii) & \|dg_x\| \cdot \|x\| & = & \mathcal{O}(\|x\|) \\ (iv) & \|dg_x\| \cdot \|x\| & \leq & \frac{1}{4}, \end{array}$$

where $\mathcal{O}$ is the usual "big-O" Landau symbol. Then the map

$$\begin{array}{rccc} f: & U & \to & U \\ & x & \mapsto & \exp\left(g(x)\right) \cdot x \end{array}$$

is a C^1-diffeomorphism on U and it is of class C^m on $U \setminus \{0\}$.

Proof: As f is a composition of C^m maps on $U \setminus \{0\}$ and a composition of C^0 maps on U, it inherits those smoothness properties. Hence it suffices to show that f is bijective and df_x has full rank for all $x \in U$ and admits an extension to $x = 0$, which also has full rank. First we observe

$$df_x(v) = \exp(g(x)) \cdot v + d\exp_{g(x)} \circ dg_x(v) \cdot x$$

for $x \in U \setminus \{0\}$. By the third requirement, the second term vanishes at $x = 0$ and hence df_x can be continuously extended to $x = 0$ by $df_0 = \mathrm{id}$, which coincides with df_0 if calculated directly. The second and fourth requirement then guarantees that f is a diffeomorphism. For any $x \in U$ we have

$$\begin{aligned} \|df_x(v)\| &\geq \|\exp(g(x)) \cdot v\| - \left\|d\exp_{g(x)} \circ dg_x(v) \cdot x\right\| \\ &\geq \|v\| - \left\|d\exp_{g(x)}\right\| \cdot \|dg_x\| \cdot \|v\| \cdot \|x\| \\ &\geq \frac{1}{2}\|v\|. \end{aligned}$$

Consequently f is a local C^1 diffeomorphism.

For f being a global diffeomorphism on U it remains to show that it is bijective. We recall that $\exp(g(x)) \in \{\mathrm{id}\} \times SO(n-1) \subset SO(n)$ leaves the first component and norms of $x = (x^0, \ldots, x^{n-1}) \in U$ and $(x^1, \ldots, x^{n-1})$ untouched. Hence it suffices to show that f is bijective on spheres $S^{n-2}_{c_1,c_2}$ defined in the lemma. The restriction of f to such spheres still is a local diffeomorphism, hence is a local homeomorphism and hence by Lemma 5.3.2 is bijective on such spheres. ■

5.3.1. Diffeomorphisms on S^n

Throughout the following subsection we will consider the sphere S^n to be a submanifold of $\mathbb{R}^{n+1}$. Furthermore we use the induced topology. In particular for $x \in S^n$ the ball $B_r(x)$ is the ball with radius r in $\mathbb{R}^{n+1}$ restricted to the sphere.

Lemma 5.3.2. *Any local homeomorphism $f : S^n \to S^n$ with $n \geq 2$ is a global homeomorphism.*

Proof: By [DC76, Proposition 5.6.1] f is a covering map, hence provides a universal cover of the sphere and consequently is a global homeomorphism. ■

Diffeomorphisms Admissibly Close to id_{S^n}

Assume $f : S^n \to S^n$ to be a C^m-diffeomorphism that is close to the identity in a useful sense. The aim of the section is to show that there is a C^m-map $g : S^n \to \underline{\mathfrak{so}}(n+1)$ such that $f(x) = \exp(g(x)) \cdot x$.

Lemma 5.3.3. *Let $f : S^n \to S^n$ be a C^m-diffeomorphism. Then there is neighbourhood $U(S^n) \subset \mathbb{R}^{n+1}$ of the sphere and an C^m-extension $F : U(S^n) \to \mathbb{R}^{n+1}$ to a neighbourhood $U(S^n)$ of the sphere such that*

$$\begin{array}{ll} (i) & \operatorname{rank} dF_x = n+1 \textit{ for } x \in S^n \\ (iia) & F(x) = f(x) \textit{ for all } x \in S^n \\ (iib) & \|F(x)\| \neq \|x\| \textit{ for all } x \in U(S^n) \setminus S^n \\ (iii) & dF_x(x) = 2f(x) \textit{ for all } x \in S^n. \end{array}$$

In the last line $f(x)$ is interpreted as a vector in the tangent space of $\mathbb{R}^{n+1}$ by canonic identification $T_x\mathbb{R}^{n+1} \simeq \mathbb{R}^{n+1}$.

Proof: We define the extension by

$$F(x) := (2\|x\| - 1) f\left(\frac{x}{\|x\|}\right).$$

This map is of class C^m as long as $x \neq 0$. We then have by definition $F(x) = f(x)$ for $x \in S^n$ and therefore on the sphere we find $dF_x(v) = df_x(v)$ for $v \in T_xS^n$. Since f was assumed to be a diffeomorphism, we have $dF_x(T_xS^n) = T_{f(x)}S^n$. Moreover, for $x \in S^n$ and $v = x$ we may calculate

$$\begin{aligned} dF_x(x) &= \left.\frac{d}{dt}\right|_{t=0} (F(x+tx)) \\ &= 2\|x\| f\left(\frac{x}{\|x\|}\right). \end{aligned}$$

This proves (i), (ii)a and (iii). If we restrict the domain of F to points with $\|x\| > 0.5$, then $\|F(x)\| = 2\|x\| - 1$ and hence $\|F(x)\| = \|x\|$ if and only if $x \in S^n$, which proves statement (ii)b. ■

Definition 5.3.4. Such an extension will be called *sphere-preserving extension* of f.

We will now say what it means for a map f to be sufficiently close to the identity. For a map $h : S^n \to \mathbb{R}^n$ we make the notation $\|h\|_{\infty,S^n} := \sup_{x\in S^n} \|h(x)\|_{\mathbb{R}^{n+1}}$ and get the following definition.

Definition 5.3.5. A C^m diffeomorphism $f : S^n \to S^n$ will be said to be *admissibly close* to the identity if there is an $\epsilon < 1$ and a very convex neighbourhood $U(\mathbb{1}) \subset SO(n+1)$ such that

(i) $U_\epsilon(\mathbb{1}) := \{A \in SO(n+1) \mid \|A - \mathbb{1}\| < \epsilon\}$ is a subset of $U(\mathbb{1})$ and

(ii) $\|f - \mathrm{id}\|_{\infty,S^n} < \epsilon$.

We denote with $\mathfrak{U}(0)$ the corresponding neighbourhood in $\underline{\mathfrak{so}}(n+1)$. In particular $\exp : \mathfrak{U}(0) \to U(\mathbb{1})$ is a diffeomorphism on that neighbourhood.

Corollary 5.3.6. *The first observation is that if $f : S^n \to S^n$ is admissibly close to the identity, then with the above notation for each $x \in S^n$ there is a $g \in \mathfrak{U}(0)$ such that $f(x) = \exp(g) \cdot x$. The second condition implies $\|f(x) - x\| < \epsilon$ for all $x \in S^n$.*

In the first part of this section we will construct local maps on the sphere with values in $\mathfrak{U}(0)$. Very convexity will then be used in the second part of this section to get a global map by gluing the local maps.

Definition 5.3.7. Let $f : S^n \to S^n$ be a map admissibly close to the identity (with respect to ϵ) and let $U(\mathbb{1})$ be the corresponding *very convex* neighbourhood. Consider $F : U(S^n) \to \mathbb{R}^{n+1}$ to be a *sphere-preserving extension* of f (see Definition 5.3.4). We then define

$$\begin{array}{rccc} \tilde{G}: & U(S^n) \times \mathfrak{U}(0) & \to & \mathbb{R}^{n+1} \\ & (x,g) & \mapsto & F(x) - \exp(g) \cdot x \end{array}$$

and

$$\tilde{M} := \tilde{G}^{-1}(0).$$

We recall the fact that F was constructed such that it is a diffeomorphism on $U(S^n)$ and such that $\|F(x)\| = \|x\|$ if and only if $x \in S^n$. Hence we have the following lemma.

Lemma 5.3.8. *With the assumptions of the last definition, $\tilde{M} \subset \mathbb{R}^{n+1} \times \underline{\mathfrak{so}}(n+1)$ is an embedded C^m-submanifold of dimension $\dim(\tilde{M}) = \frac{n(n+1)}{2}$. Moreover, we find $\tilde{M} \subset S^n \times \mathfrak{U}(0)$.*

Proof: Clearly x is an element of $\tilde{M}$ if and only if $F(x) = \exp(g) \cdot x$. The first observation is that this necessarily requires $\|F(x)\| = \|\exp(g) \cdot x\| = \|x\|$. By construction of F the latter equation holds if and only if $x \in S^n$. This immediately gives $\tilde{M} \subset S^n \times \mathfrak{U}(0)$.

It suffices to show that 0 is a regular value of $\tilde{G}$. The differential $d\tilde{G}_{(x,g)} : \mathbb{R}^{n+1} \times T_g\underline{\mathfrak{so}}(n+1) \to \mathbb{R}^{n+1}$ is given by

$$\begin{aligned} d\tilde{G}_{(x,g)}(v_x, v_g) &= \frac{d}{dt}\Big|_{t=0} \tilde{G}(x + tv_x, g + tv_g) \\ &= dF_x(v_x) - (d\exp)_g(v_g) \cdot x - \exp(g) \cdot v_x. \end{aligned} \tag{5.18}$$

By restricting to $v_x = 0$ and using (1.88) for the last step we find for $x \in S^n$

$$\begin{aligned} d\tilde{G}_{(x,g)}\left(0, \underline{\mathfrak{so}}(n+1)\right) &= d\exp_g\left(\underline{\mathfrak{so}}(n+1)\right) \cdot x \\ &= T_{\exp(g)} SO(n+1) \cdot x \\ &= T_{\exp(g)\cdot x} S^n \\ &= T_{f(x)} S^n. \end{aligned}$$

Now we choose $v_x = x$ using the usual identification of $\mathbb{R}^{n+1}$ and its tangent space and $v_g = 0$. Then $d\tilde{G}_{(x,g)}(x,0) = dF_x(x) - \exp(g) \cdot x = dF_x(x) - f(x) = 2f(x) - f(x) = f(x)$. Considered as a vector in tangent space, $f(x)$ spans the normal space $N_{f(x)}S^n$ in $f(x)$. As a consequence $N_{f(x)}S^n \subset \operatorname{im}\left(dG_{(x,g)}\right)$. Together with the last calculation this gives $\operatorname{im}(dG_{x,g}) = N_{f(x)}S^n \otimes T_{f(x)}S^n \simeq \mathbb{R}^{n+1}$. Hence $d\tilde{G}_{(x,g)}$ is a surjective map for each $(x,g) \in \tilde{M}$ and the regular value theorem can be applied. $\tilde{M}$ then is a $\frac{n(n+1)}{2}$ dimensional submanifold of $U(S^n) \times \mathfrak{U}(0)$. ∎

The last lemma basically holds for any map on S^n. In the following we will have the more restrictive demand on $\tilde{M}$ to be surjectively projected to the sphere. First of all $T_{(x,g)}\tilde{M}$ clearly is a hypersurface in $T_xS^n \times T_g\underline{\mathfrak{so}}(n+1)$. But at this point we have no information on its orientation in the ambient tangent space. For the subsequent treatment it is of high importance that there is no vector in $T_{(x,g)}\tilde{M}$ that is an element of $\{0\} \times T_g\underline{\mathfrak{so}}(n+1)$. We will show that $d\pi_{S^n}$ has rank n for all $p \in \tilde{M}$. This property is provided by the following lemma.

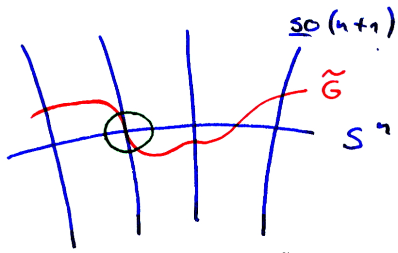

Figure 6.: The orientation of $\tilde{G}$ w.r.t. $S^n \times \mathfrak{U}(0)$ marked by the circle will not occur.

Lemma 5.3.9. *The projection map $\pi_{S^n} : \tilde{M} \ni (x,g) \mapsto x \in S^n$ is a surjective submersion.*

Proof: Surjectivity of π_{S^n} is provided by the construction method of $\tilde{M}$. Consider $x \in S^n$, then, since f is admissibly close to the identity, we have $\|f(x) - x\| < \epsilon$. Hence there is a rotation $A \in SO(n+1)$ such that $A \cdot x = f(x)$ and $\|A - \mathbb{1}\| < \epsilon$. Therefore $A \in U(\mathbb{1})$ and with $g := \exp^{-1}(A)$ we find $(x, g) \in \tilde{M}$ and hence $x \in \mathrm{im}(\pi)$.

By definition we have $T_{(x,g)}\tilde{M} \subset T_x S^n \times T_g \underline{\mathfrak{so}}(n+1)$. The regular value theorem in particular gives $T_{(x,g)}\tilde{M} = \ker\left(d\tilde{G}_{(x,g)}\right)$. To show that $d\pi$ has full rank for all $(x, g) \in \tilde{M}$ it suffices to show that the dimension of its kernel is $\dim(\tilde{M}) - \dim(S^n) = \frac{n(n-1)}{2}$. For $(v_x, v_g) \in \ker\left(d\tilde{G}_{(x,g)}\right)$ we have $(v_x, v_g) \in \ker(d\pi_{(x,g)})$ if and only if $v_x = 0$. Hence by (5.18) the kernel is spanned by all $(0, v_g)$ with $(0, v_g) \in \ker\left(d\tilde{G}_{(x,g)}\right)$, which is equivalent to requiring $(d\exp)_g\,(v_g) \cdot x = 0$. The differential $(d\exp)_g$ maps to $T_{\exp(g)}SO(n+1)$ and so by (1.87) there is an $X \in \underline{\mathfrak{so}}(n+1)$ such that $(d\exp)_g\,(v_g) = \exp(g) \cdot X$. Now $\exp(g) \in SO(n+1)$ is an isomorphism on $\mathbb{R}^{n+1}$ and so by (1.89) the equation $\exp(g) \cdot X \cdot x = 0$ is equivalent to requiring $X \in \underline{\mathfrak{stab}}(x)$. Finally, we use that $\exp(g)$ is an element of a convex neighbourhood of $\mathbb{1}$ and hence $(d\exp)_g$ is an isomorphism. As a consequence the previous considerations can be summarised as

$$(v_x, v_g) \in \ker(d\pi_{(x,g)}) \iff \begin{cases} v_x = 0 \\ v_g \in (d\exp)_g^{-1}\left(\exp(g) \cdot \underline{\mathfrak{stab}}(x)\right). \end{cases}$$

Hence $\ker(d\pi_{(x,g)}) \simeq \underline{\mathfrak{stab}}(x) \simeq \underline{\mathfrak{so}}(n)$, which can be used to calculate the dimension of the image

$$\begin{aligned} \dim \mathrm{im}\left(d\pi_{(x,g)}\right) &= \dim T_{(x,g)}\tilde{M} - \dim\ker(d\pi_{(x,g)}) \\ &= \frac{(n+1)n}{2} - \frac{n(n-1)}{2} = n \\ &= \dim T_x S^n. \end{aligned}$$

Therefore $d\pi_{(x,g)}$ is surjective for each $(x, g) \in \tilde{M}$. ∎

Lemma 5.3.10. *For all $x \in S^n$ there is a neighbourhood $U(x) \subset S^n$ and a map $g : U(x) \to \underline{\mathfrak{so}}(n+1)$ such that*

(i) $(y, g(y)) \in \tilde{M}$ *for all* $y \in U(x)$.
(ii) g *is* C^m*-smooth.*

Proof: We first observe $\tilde{M} \subset S^n \times \underline{\mathfrak{so}}(n+1) \subset \mathbb{R}^{n+1} \times \underline{\mathfrak{so}}(n+1)$. By Lemma 5.3.8 $\tilde{M} \subset \mathbb{R}^{n+1} \times \underline{\mathfrak{so}}(n+1)$ is an embedded submanifold and by using the natural inclusion, $S^n \times \underline{\mathfrak{so}}(n+1) \subset \mathbb{R}^{n+1} \times \underline{\mathfrak{so}}(n+1)$ is a embedded submanifold itself. Hence[4] $\tilde{M}$ is an embedded submanifold of $S^n \times \underline{\mathfrak{so}}(n+1)$ $\tilde{M}$ of class C^m.

Now let $p \in \tilde{M} \subset S^n \times \underline{\mathfrak{so}}(n+1)$ be an arbitrary point and define $d := \frac{n(n-1)}{2}$. As we have seen before, $\tilde{M}$ is a $\frac{n(n+1)}{2} = n + d$ dimensional C^m submanifold of $S^n \times \underline{\mathfrak{so}}(n+1)$. Hence there is a special C^m-submanifold chart $\varphi : U \subset S^n \times \underline{\mathfrak{so}}(n+1) \to \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^n$ for a neighbourhood of $p \in \tilde{M}$ in $S^n \times \underline{\mathfrak{so}}(n+1)$ such that

$$\begin{aligned} &\varphi(p) = 0 \\ &\varphi(q) \in \mathbb{R}^{n+d} \times \{0\} \text{ for all } q \in \tilde{M} \cap U. \end{aligned}$$

Only the last $\mathbb{R}^n$ component in the decomposition $\mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^n$ is relevant at the moment. It parametrises the space away from $\tilde{M}$. The remaining decomposition of $\mathbb{R}^{\frac{n(n+1)}{2}}$ will be justified next. We consider the map

$$\vartheta := \pi_{S^n} \circ \varphi^{-1}\Big|_{\mathbb{R}^{n+d} \times \{0\}} : \mathbb{R}^{n+d} \times \{0\} \cap \varphi(U) \to S^n.$$

4 The conclusion is provided by the fact that for manifolds A, B and C and an embedding $B \overset{\iota}{\hookrightarrow} C$, the diagram $A \xrightarrow{f} B \overset{\iota}{\hookrightarrow} C$ provides an embedding only if f is an embedding.

The image $\varphi^{-1}\left(\mathbb{R}^{n+d} \times \{0\} \cap \varphi(U)\right)$ is a neighbourhood of p in $\tilde{M}$. By Lemma 5.3.9 π_{S^n} is a surjective submersion and hence the composition $\vartheta = \pi_{S^n} \circ \varphi^{-1}$ is a surjective submersion too. Without loss of generality we will assume φ to have been chosen such that

$$\ker d\vartheta_{\varphi(p)=0} = \{0\} \times \mathbb{R}^d \times \{0\}$$

in the first place. This may be achieved by a rotation in $\mathbb{R}^{n+d}$. With that choice $\left(\vartheta|_{\mathbb{R}^n\times\{0\}\times\{0\}}\right)_0$ consequently has full rank and the restricted map

$$\tilde{\vartheta} := \vartheta|_{\mathbb{R}^n\times\{0\}\times\{0\}} : \mathbb{R}^n \times \{0\} \times \{0\} \to S^n$$

is a diffeomorphism for some neighbourhood $U_0 \subset \mathbb{R}^n \times \{0\} \times \{0\}$ of the origin. In particular this makes π_{S^n} a diffeomorphism, which maps $\phi^{-1}(U_0)$ to the sphere[5]. In addition we have $\tilde{\vartheta}(0) = p$. We now define the map g as follows

$$\begin{array}{rccc} g: & \tilde{\vartheta}(U_0) & \to & \underline{\mathfrak{so}}(n+1) \\ & y & \mapsto & \pi_{\underline{\mathfrak{so}}(n+1)} \circ \varphi^{-1} \circ \tilde{\vartheta}^{-1}(y). \end{array}$$

It can be visualised by the diagram

$$S^n \xrightarrow{\tilde{\vartheta}^{-1}} \mathbb{R}^n \times \{0\} \times \{0\} \xrightarrow{\varphi^{-1}} \tilde{M} \subset S^n \times \underline{\mathfrak{so}}(n+1) \xrightarrow{\pi_{\underline{\mathfrak{so}}(n+1)}} \underline{\mathfrak{so}}(n+1).$$

The map g is of class C^m since it is a composition of such maps. Now it remains to ensure that we really have $(y, g(y)) \in \tilde{M}$ for all $y \in \tilde{\vartheta}(U_0)$. Since g provides the projection of $\varphi^{-1} \circ \tilde{\vartheta}^{-1}(y)$ to the $\underline{\mathfrak{so}}(n+1)$ component of $\tilde{M}$, we have to show that the S^n component of $\varphi^{-1} \circ \tilde{\vartheta}^{-1}(y)$ is y. This is done by the following calculation

$$\begin{aligned} \pi_{S^n} \circ \varphi^{-1} \circ \tilde{\vartheta}^{-1}(y) &= \left(\pi_{S^n} \circ \varphi^{-1}\right) \circ \left(\pi_{S^n} \circ \varphi^{-1}\Big|_{\mathbb{R}^n\times\{0\}\times\{0\}}\right)^{-1}(y) \\ &= y. \end{aligned}$$

In other words, the proof used that all maps in the following diagram are at least C^m-smooth and the maps in the lower part are invertible. Then g on the right-hand side exists as the composition of maps represented by solid lines. It is well defined, since π_{S^n} is invertible in this diagram.

$$\begin{tikzcd}
 & & \underline{\mathfrak{so}}(n+1) \\
\mathbb{R}^n \times \{0\} \times \{0\} & S^n \times \underline{\mathfrak{so}}(n+1) & \\
\cup & \cup & \\
U_0 \arrow[r, leftrightarrow, "\varphi"'] \arrow[rr, leftrightarrow, bend right, "\tilde{\vartheta}"'] & \varphi^{-1}(U_0) \arrow[uuur, "\pi_{\underline{\mathfrak{so}}(n+1)}"'] \arrow[r, leftrightarrow, dashed, "\pi_{S^n}"'] & S^n \arrow[uuu, "g"']
\end{tikzcd}$$

■

Aim of the remaining part of this section is to make this local result a global one.

Lemma 5.3.11. *Let $x \in S^n$ be a point at the sphere, $U(\mathbb{1}) \subset SO(n+1)$ a convex neighbourhood of the identity, $\mathfrak{U}(0) := \exp^{-1}(U(\mathbb{1}))$, $h \in \mathfrak{U}(0)$ and $\tilde{F} := \{g \in \mathfrak{U}(0) \mid \exp(g)x = \exp(h)\cdot x\}$ then*

$$\begin{array}{ll} (i) & \exp\left(\tilde{F}\right) = \left(\exp(h)\cdot \mathrm{stab}(x)\right) \cap U(\mathbb{1}). \\ (ii) & \tilde{F} \text{ is connected.} \end{array}$$

$stab(x) \subset SO(n+1)$ is the stabiliser subgroup of x. In particular claim (i) persists even if h is replaced by any other element of $\tilde{F}$.

5 Bijectivity of π_{S^n} is provided by the observation that $\pi_{S^n} \circ \varphi^{-1}|_{\mathbb{R}^n\times\{0\}\times\{0\}}$ is bijective.

Proof: First we observe that since h is an element of $\tilde{F}$, the latter is not an empty set.

To proof part (i), we will show the correctness of the two inclusions of the equation. For the inclusion $\exp\left(\tilde{F}\right) \subset \exp(h) \cdot \operatorname{stab}(x) \cap U(\mathbb{1})$ consider an element $g \in \tilde{F}$. Then by definition $\exp(-h)\exp(g) \cdot x = x$. Hence $A := \exp(-h)\exp(g)$ stabilises x and $\exp(h) \cdot A = \exp(g) \in U(\mathbb{1})$. The inclusion follows.

Now let $A \in SO(n+1)$ be an x-stabilising matrix such that $\exp(h) \cdot A =: A' \in U_{\mathbb{1}}$. Then $g := \exp^{-1}(A') \in \mathfrak{U}(0)$ is well defined and it holds $\exp(g) \cdot x = \exp(h) \cdot A \cdot x = \exp(h) \cdot x$. Consequently we have $g \in \tilde{F}$ and hence $\exp(\tilde{F}) \supset \exp(h) \cdot \operatorname{stab}(x) \cap U_{\mathbb{1}}$. This gives the opposite inclusion and claim (i) follows.

For the second part consider $h_1, h_2 \in \tilde{F} \subset \mathfrak{U}(0)$ such that $\exp(h_1) \cdot x = \exp(h_2) \cdot x = y$. By Lemma 1.3.4(ii) there is a map $\eta : I = [0,1] \to \mathfrak{U}(0)$ such that $\eta(0) = h_1$, $\eta(1) = h_2$ and $\exp(\eta(t)) \cdot x = y$ for all $t \in I$. Hence $\eta(t) \in \tilde{F}$ for all $t \in I$ and $\tilde{F}$ consequently is connected. ■

Corollary 5.3.12. *From statement (i) we conclude* $\tilde{F} = \exp^{-1}\left(\exp(h) \cdot \operatorname{stab}(x) \cap U(\mathbb{1})\right)$.

Definition 5.3.13. Let $f : S^n \to S^n$ be a map admissibly close to the identity, $U(\mathbb{1})$ the corresponding very convex neighbourhood, $\mathfrak{U}(0) := \exp^{-1}\left(U(\mathbb{1})\right)$ and $\tilde{M}$ the submanifold constructed in Definition 5.3.7. Then by Lemma 5.3.9 the projection $\tilde{\pi}_{S^n} : \tilde{M} \to S^n$ is a surjective submersion and hence provides a fibration of $\tilde{M}$. For the fibres we define

$$\{x\} \times \tilde{F}_x := \tilde{\pi}^{-1}(x)$$

Lemma 5.3.14. *With the requirements of the previous definition,* $\tilde{F}_x \subset \underline{\mathfrak{so}}(n+1)$ *is a connected set for each* $x \in \tilde{M}$. *It is explicitly given by*

$$\tilde{F}_x := \{g \in \mathfrak{U}(0) \mid \exp(g)x = f(x)\}.$$

Proof: Since f is admissibly close to the identity there is an $h \in \mathfrak{U}(0)$ such that $f(x) = \exp(h) \cdot x$. Consequently $\tilde{F}_x$ can be written as

$$\tilde{F}_x = \{g \in \mathfrak{U}(0) \mid \exp(g) \cdot x = \exp(h) \cdot x\}$$

and therefore it is of the form needed for applying Lemma 5.3.11. The claim follows. ■

In particular by Corollary 5.3.12 the fibres are explicitly given by $\tilde{F}_x = \exp^{-1}\left(\exp(h) \cdot \operatorname{stab}(x) \cap U(\mathbb{1})\right)$.

Lemma 5.3.15. *Let* $\mathfrak{U}(0)$ *be the preimage of a very convex neighbourhood* $U(\mathbb{1})$. *Let* g, h *be elements in* $\mathfrak{O}$, *where either* $\mathfrak{O} = \mathfrak{U}(0)$ *or* $\mathfrak{O} = \tilde{F}_x = \exp^{-1}\left(\exp(h) \cdot \operatorname{stab}(x) \cap U(\mathbb{1})\right)$ *for some* $x \in S^n$. *Further we define for the moment* $G \in SO(n+1)$, $\tilde{h} \in \underline{\mathfrak{so}}(n+1)$ *and a curve* $\gamma : [0,1] \to SO(n+1)$ *by*

$$G := \exp(g) \qquad h =: \exp^{-1}\left(G \cdot \exp\left(\tilde{h}\right)\right)$$
$$\gamma(t) := \exp^{-1}\left(G \cdot \exp\left(t\tilde{h}\right)\right).$$

Then $\gamma(t) \in \mathfrak{O}$ *for all* $t \in [0,1]$ *and* $\gamma(0) = g$, $\gamma(1) = h$.

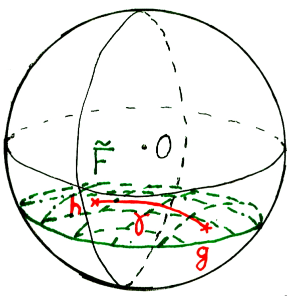

Figure 7.: Preimage of geodesics in $\tilde{F} \subset \mathfrak{U}(0)$.

We will point out the main ideas of this lemma. $U(\mathbb{1})$ is a (very) convex neighbourhood, this fact provides that it is a normal neighbourhood of each of its points. So it is clear that there is a geodesic between any two points in $U(\mathbb{1})$. The lemma gives an explicit form for the preimage of that geodesic under the exponential map in $\mathbb{1} \in SO(n+1)$. The second fact provided by the lemma is that the geodesic connecting two points in the subset $\tilde{F}$ completely belongs to $\tilde{F}$.

Proof: We observe that by definition G and $\exp(h)$ are elements of the very convex neighbourhood $U(\mathbb{1})$. Hence by definition of very convexity $\tilde{h} = \exp^{-1}\left(G^{-1} \cdot \exp(h)\right)$ is well defined even if it is not in $\mathfrak{U}(0)$. The two equations $\gamma(0) = g$ and $\gamma(1) = h$ then follow from direct calculations.

First assume $\mathfrak{O} = \mathfrak{U}(0)$. We now calculate $\exp(\gamma(t))$ and use the explicit form of $\exp_G : T_G SO(n+1) \to SO(n)$ for the last step

$$\begin{aligned} \exp(\gamma(t)) &= G \cdot \exp\left(t\tilde{h}\right) \\ &= G \cdot \exp\left(G^{-1} t G \tilde{h}\right) \\ &= \exp_G\left(tG\tilde{h}\right). \end{aligned}$$

The definition in the requirements of the lemma then provides $\tilde{h} \in \underline{\mathfrak{so}}(n+1)$. By (1.87) this gives $G\tilde{h} \in T_G SO(n+1)$. Now clearly $\exp \circ \gamma$ is the geodesic connecting $\exp(h)$ and $\exp(g)$, both elements of the very convex neighbourhood $U(\mathbb{1})$. Hence $\gamma(t) \in \mathfrak{U}(0) = \mathfrak{O}$ for all $t \in [0,1]$.

Now consider $\mathfrak{O} = \tilde{F} \subset \mathfrak{U}(0)$. By Lemma 5.3.11 we have $\exp\left(\tilde{F}\right) = G \cdot \operatorname{stab}(x) \cap U_{\mathbb{1}}$ and hence $\exp\left(\tilde{h}\right) \in \operatorname{stab}(x)$. The preimage of $\operatorname{stab}(x)$ is generated by a linear subspace of $\underline{\mathfrak{so}}(n+1)$ isomorphic to $\underline{\mathfrak{so}}(n)$. Hence $\exp\left(t\tilde{h}\right)$ still is in $\operatorname{stab}(x)$ and hence $G \cdot \exp\left(t\tilde{h}\right) = \exp(\gamma(t)) \in \exp\left(\tilde{F}\right)$ which proves $\gamma(t) \in \tilde{F}$ for all $t \in [0,1]$. ∎

As mentioned before the fibres $\tilde{F}_x$ have the form needed for applying the last lemma for each $x \in S^n$. This provides for any two points in $\tilde{F}_x$ a smooth curve γ, which is completely within the fibre and connects the two points.

Remark 5.3.16. By 5.3.10 for each $x \in S^n$, there is an open ball $B_x \subset S^n$ and a C^m-map $g_x : B_x \to \mathfrak{U}(0)$ such that $(y, g_x(y)) \in \tilde{M}$ for all $y \in B_x$. The notation of a ball refers to open balls in $\mathbb{R}^{n+1}$, which have been restricted to the sphere. Since the sphere is compact, there is a finite cover of such balls that we will denote by

$$S^n = \bigcup_{i=1}^{k} B_i.$$

Further the corresponding map will be $g_i : B_i \to \mathfrak{U}(0)$.

We will now provide a method to glue the different maps g_i to a global map on S^n. As a preparation the following lemma is needed. It also acts as definition for the balls $\tilde{B}_i$.

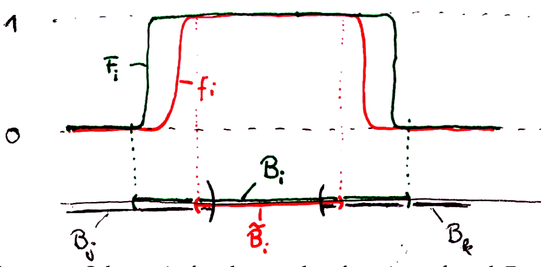

Figure 8.: Schematic for the overlap functions f_i and F_i on the ball B_i.

Lemma 5.3.17. *Let $S^n = \bigcup_{i=1}^k B_i$ be a finite cover of open balls B_i. Then for each $i \in \{1, \dots, m\}$ there is a smaller ball $\tilde{B}_i \subset B_i$ and smooth maps $f_i : S^n \to [0,1]$ and $F_i : S^n \to [0,1]$ such that*

$$S^n = B_1 \cup \dots \cup B_{i-1} \cup \tilde{B}_i \cup B_{i+1} \cup \dots \cup B_k$$

still is an open cover, $\overline{\operatorname{supp}(F_i)} \subset B_i$ is compact and

$$\operatorname{supp}(f_i) \subset \operatorname{supp}(F_i) \qquad f_i|_{\tilde{B}_i} \equiv 1 \qquad F_i|_{\operatorname{supp}(f_i)} \equiv 1.$$

A sketch of the basic proof can be found in the appendix.

Corollary 5.3.18. *Applying this lemma successively to all B_i we get a cover $S^n = \bigcup_{i=1}^k \tilde{B}_i$ and tuple $(B_i, \tilde{B}_i, g_i, f_i, F_i)$ with the above properties. The f_i fulfil*

$$f_i|_{\tilde{B}_i} \equiv 1 \qquad f_i|_{S^n \setminus B_i} \equiv 0$$

and are smooth in between. Hence we have $f_i \cdot g_i : B_i \to \mathfrak{U}(0)$ to be a C^m map and to fulfil

$$(x, f_i(x) \cdot g_i(x)) =|_{\tilde{B}_i} (x, g_i(x)) \in \tilde{F}_x \qquad f_i \cdot g_i|_{S^n \setminus B_i} \equiv 0.$$

Therefore $f_i \cdot g_i$ can be extended to a function on S^n by setting $(f_i \cdot g_i)(x) = 0 \in \mathfrak{U}(0)$ outside B_i. The same argument works for $(F_i \cdot g_i)(x)$.

We will now summarise some facts in the following lemma.

Lemma 5.3.19. *Let $(U_1, \tilde{U}_1, g_1, f_1)$ and $(U_2, \tilde{U}_2, g_2, f_2, F)$ be tuples with open sets U_i, $\tilde{U}_i \subset S^n$, smooth maps $f_1, f_2, F : S^n \to [0,1]$ and C^m-maps $g_i : U_i \to \mathfrak{U}(0)$ such that following properties hold for $i \in \{1,2\}$*

(i) $U_i, \tilde{U}_i$ are open sets with $\tilde{U}_i \subset \overline{\operatorname{supp}(f_i)} \subset U_i$.

(ii) f_i fulfils

$$f_i|_{\tilde{U}_i} \equiv 1 \qquad\qquad f_i|_{S^n \setminus U_i} \equiv 0.$$

(iii) $(x, g_1(x)) \in \tilde{M}$ for all $x \in \tilde{U}_1$.

(iv) $(x, g_2(x)) \in \tilde{M}$ for all $x \in U_2$, and

$$F|_{\operatorname{supp} f_2} \equiv 1 \qquad\qquad F|_{S^n \setminus U_i} \equiv 0.$$

Then for $U := U_1 \cup U_2$ and $\tilde{U} := \tilde{U}_1 \cup \tilde{U}_2$ there are a smooth map $f : S^n \to [0,1]$ and a C^m map $g : U \to \mathfrak{U}$ such that the tuple $(U, \tilde{U}, g, f)$ fulfils properties (i) to (iii) of this lemma.

As we can see, this lemma provides an inductive step. The map g, whose existence is claimed, can be used to act as a new g_1.
Proof: We define the maps $f, \phi : S^n \to [0,1]$ by

$$\phi(x) := (1 - f_2(x))f_1(x) \qquad\qquad f(x) := f_2(x) + \phi(x)$$

and

$$\begin{aligned} G(x) &:= \exp\left(\left(F \cdot g_2\right)(x)\right) \\ \tilde{h}(x) &:= \exp^{-1}\left[\left(G(x)\right)^{-1} \cdot \exp\left(\left(f_1 \cdot g_1\right)(x)\right)\right] \\ \gamma(t) &:= \exp^{-1}\left[G(x) \cdot \exp\left(t\tilde{h}(x)\right)\right] \\ g(x) &:= \gamma\left(\phi(x)\right). \end{aligned}$$

If written more explicitly we see $f(x) = f_2(x) \cdot 1 + (1 - f_2(x))f_1(x)$ to map to a value between 1 and $f_2(x)$ and hence also maps to the interval $[0,1]$. Moreover, by definition $f(x) = 1$ if $f_1(x) = 1$ or $f_2(x) = 1$ and $f(x) = 0$ if $f_1(x) = 0 = f_2(x)$. Hence property (ii) is fulfilled. Moreover, $\operatorname{supp}(f) = \operatorname{supp}(f_1) \cup \operatorname{supp}(f_2)$ such that property (i) is fulfilled immediately.

Now we observe that $\phi(x) \in [0,1]$ and $(f_1 \cdot g_1)(x), (F \cdot g_2)(x) \in \mathfrak{U}(0)$. Hence Lemma 5.3.15 can be applied and we get $g(x) \in \mathfrak{U}(0)$. We now have to study three cases. First assume $x \in \tilde{U}_2$. Then we have $f_2(x) = 1$ and $\phi(x) = 0$. Hence $g(x) = \gamma(0) = g_2(x) \in \tilde{F}_x$ such that $(x, g(x)) \in \tilde{M}$. Now assume $x \in \tilde{U}_1 \setminus \tilde{U}_2$ and $x \in \operatorname{supp}(f_2)$. Then by using the requirements of the lemma we have $F(x) = f_1(x) = 1$ and hence $(f_1 \cdot g_1)(x), (F \cdot g_2)(x) \in \tilde{F}_x$. Applying Lemma 5.3.15 gives $g(x) \in \tilde{F}_x$. Now consider $x \in \tilde{U}_1 \setminus \tilde{U}_2$ and $f_2(x) = 0$. Then we have $f_1(x) = 1$ and $\Phi(x) = 1$ such that $g(x) = \gamma(1) = g_1(x) \in \tilde{F}_x$. Summarising the three cases we get $(x, g(x)) \in \tilde{M}$ for all $x \in \tilde{U}$ such that (iii) is fulfilled. ■

Proposition 5.3.20. *There is a C^m map $g : S^n \to \underline{\mathfrak{so}}(n+1)$ such that*

$$(x, g(x)) \in \tilde{M}$$

for all $x \in S^n$.

Proof: By Remark 5.3.16 there is an open cover

$$S^n = \bigcup_{i=1}^{k} B_i$$

of open balls and maps $g_i : B_i \to \mathfrak{U}(0)$ such that $(x, g_i(x)) \in \tilde{M}$ for x in the domain of g_i. By Corollary 5.3.18 we then get tuples $(B_i, \tilde{B}_i, g_i, f_i, F_i)$ that fulfil properties (i),(ii) and (iv) of Lemma 5.3.19. The proof will now be done by some inductive method.

The tuple $(B_1, \tilde{B}_1, g_1, f_1)$ fulfils properties (i) to (iii). Hence Lemma 5.3.19 can be applied to $(B_1, \tilde{B}_1, g_1, f_1)$ and $(B_1, \tilde{B}_1, g_1, f_1, F_1)$ such that there is a new tuple, which will be denoted $(U_2, \tilde{U}_2, \tilde{g}_2, \tilde{f}_2)$, which is compatible with properties (i) to (iii) and such that

$$\tilde{U}_2 = \bigcup_{i=1}^{2} \tilde{B}_i.$$

Now applying Lemma 5.3.19 to the tuple $(U_m, \tilde{U}_m, \tilde{g}_m, \tilde{f}_m)$ and $(B_{m+1}, \tilde{B}_{m+1}, g_{m+1}, f_{m+1}, F_{m+1})$ we get a new tuple $(U_{m+1}, \tilde{U}_{m+1}, \tilde{g}_{m+1}, \tilde{f}_{m+1})$ fulfilling property (i) to (iii). Moreover, we have

$$\tilde{U}_{m+1} = \left(\bigcup_{i=1}^{m} \tilde{B}_i \right) \cup \tilde{B}_{m+1} = \bigcup_{i=1}^{m+1} \tilde{B}_i.$$

The induction stops for $m = k$. The map $g := \tilde{g}_k$ then is the desired one. ∎

This section can be summarised as follows

Proposition 5.3.21. *Let $f : S^n \to S^n$ be a C^m-diffeomorphism admissibly close to the identity. Then there is a C^m-map $g : S^n \to \underline{\mathfrak{so}}(n+1)$ such that*

$$f(x) = \exp(g(x)) \cdot x.$$

We will now state some consequences of this result. First we draw our attention to the fact that if we choose ϵ small enough, then every map with $\|f - \mathrm{id}\|_{\infty, S^n} < \epsilon$ is admissibly close to the identity. This can be traced back to the fact that the very convex neighbourhood $U(\mathbb{1})$ of $\mathbb{1} \in SO(n+1)$ exists. For κ small enough, $U_{2\kappa}(\mathbb{1}) := \{A \in SO(n+1) \mid \|A - \mathbb{1}\| < 2\kappa\}$ is a subset of $U(1)$. Hence every diffeomorphism $f : S^n \to S^n$ with $\|f - \mathrm{id}\|_{\infty, S^n} < \epsilon \leq \kappa$ will be admissibly close to the identity and so has a representation of the above form. As a consequence we have the following weaker corollary.

Corollary 5.3.22. *Let $K \subset \mathrm{Diff}^m(S^n)$ be a connected component in the space of C^m diffeomorphisms of S^n with topology induced by the uniform norm. Then either for all or for no $f \in K$ there exists a C^m-map $G : S^n \to SO(n+1)$ such that $f(x) = G(x) \cdot x$.*

Proof: By Proposition 5.3.21 there is an open ball $B_\kappa(\mathrm{id}) := \{f \in \mathrm{Diff}^m(S^n) \mid \|f - \mathrm{id}\|_{\infty, S^n} < \kappa\}$ for which the claim holds. Now let $\tilde{K} \subset \mathrm{Diff}^m(S^n)$ be the subset such that for all $f \in \tilde{K}$ there exists a C^m-map $G^f : S^n \to SO(n+1)$ such that $f(x) = G^f(x) \cdot x$. It suffices to show that this set is closed and open with respect to the uniform norm. First we observe that if $f, h \in \mathrm{Diff}^m(S^n)$ with $\|f - h\|_{\infty, S^n} < \kappa$, then $\left\|f \circ h^{-1} - \mathrm{id}\right\|_{\infty, S^n} < \kappa$ since $\left\|f \circ h^{-1}(x) - x\right\| = \left\|(f - h)\left(h^{-1}(x)\right)\right\| \leq \|f - h\|_{\infty, S^n}$. Hence $f \circ h^{-1}(x)$ can be written as $G^{f \circ h^{-1}}(x) \cdot x$. Consequently if $h \in \tilde{K}$ then $f(x) = \left(f \circ h^{-1}\right)(h(x)) = G^{f \circ h^{-1}}(h(x)) \cdot G^h(x) \cdot x$ and with $G^f(x) := G^{f \circ h^{-1}}(h(x)) \cdot G^h(x)$ we have $f \in \tilde{K}$. Hence $h \in \tilde{K}$ if and only if $B_\kappa(h) \subset \tilde{K}$. For the complement we have $h \in \tilde{K}^C$ if and only if there is some $\kappa > 0$ such that $B_\kappa(h) \subset \tilde{K}^C$. Otherwise for $f \in B_\kappa(h) \cap \tilde{K} \neq \emptyset$ one also has $h \in B_\kappa(f)$ and by the previous considerations h itself will have to be an element of $\tilde{K}$, which is a contradiction. ∎

5.3.2. Diffeomorphisms on the Cylinder

The same methods used for characterising diffeomorphisms on the sphere will now be used to characterise special diffeomorphisms on the compact cylinder

$$\mathcal{Z} := [-1, 1] \times S^n \subset [-1, 1] \times \mathbb{R}^{n+1}.$$

Definition 5.3.23. A diffeomorphism $f : \mathcal{Z} \to \mathcal{Z}$ will be said to be a *sphere-preserving diffeomorphism,* if there exists an $f_{S^n} : \mathcal{Z} \to S^n$ such that $f(t,x) = (t, f_{S^n}(t,x))$. Hence $f|_{\{t\}\times S^n} : \{t\} \times S^n \to \{t\} \times S^n$ provides a diffeomorphism on the sphere for a fixed parameter $t \in [0,1]$.

Definition 5.3.24. A C^m-diffeomorphism $f : \mathcal{Z} \to \mathcal{Z}$ will said to be *admissibly close* to the identity (with respect to ϵ), if it is sphere-preserving and there is an $\epsilon < 1$ and a very convex neighbourhood $U(\mathbb{1}) \subset SO(n+1)$ such that for each $t \in [-1,1]$ the map $(S^n \ni x \mapsto f_{S^n}(t,x) \in S^n)$ is admissibly close to the identity with respect to ϵ and $U(\mathbb{1})$. The preimage of $U(\mathbb{1})$ will again be denoted $\mathfrak{U}(0) = \exp^{-1}(U(\mathbb{1}))$.

Lemma 5.3.25. *Let $f : \mathcal{Z} \to \mathcal{Z}$ be a sphere-preserving C^m-diffeomorphism. Then there is neighbourhood $U(\mathcal{Z}) \subset [-1,1] \times \mathbb{R}^{n+1}$ of the cylinder and an C^m-extension $F = (F_{\mathbb{R}}, F_{\mathbb{R}^{n+1}}) : U(\mathcal{Z}) \to [-1,1] \times \mathbb{R}^{n+1}$ such that*

$$\begin{array}{ll} (i) & \operatorname{rank} dF_p = n+2 \text{ for all } p \in \mathcal{Z} \\ (iia) & F(p) = f(p) \text{ for all } p \in \mathcal{Z} \\ (iib) & \|F_{\mathbb{R}^{n+1}}(t,x)\|_{\mathbb{R}^{n+1}} \neq \|x\|_{\mathbb{R}^{n+1}} \text{ for all } p = (t,x) \in U(\mathcal{Z}^n) \setminus \mathcal{Z}^n \\ (iii) & dF_{(t,x)}(0,x) = 2\,(0, f_{S^n}(t,x)) \text{ for all } p = (t,x) \in \mathcal{Z}, \end{array}$$

where $f_{S^n} := \pi_{\mathbb{R}^{n+1}} \circ f$. In the last line $(0, f_{S^n}(t,x))$ is interpreted as a vector in the tangent space $T_p\mathbb{R}^{n+2} \simeq \mathbb{R}^{n+2}$.

The neighbourhood in the last lemma is understood as open set with respect to the induced topology of $[-1,1] \times \mathbb{R}^{n+1}$.
Proof: First we define the neighbourhood as $U(\mathcal{Z}) := \left\{ (t,x) \in [-1,1] \times \mathbb{R}^{n+2} \,\middle|\, \|x\| > 1/2 \right\}$ and we further define the extension by

$$F(t,x) := \left(t, (2\|x\| - 1)\, f_{S^n}\left(t, \frac{x}{\|x\|}\right)\right).$$

Then the map $F^t : \left\{ x \in \mathbb{R}^{n+1} \,\middle|\, \|x\| > 1/2 \right\} \to \mathbb{R}^{n+1}$ defined by $(x \mapsto F_{\mathbb{R}^{n+1}}(t,x))$ is a sphere-preserving extension of f^t in the sense of Definition 5.3.4, where $f^t(x) := f_{S^n}(t,x)$. The Jacobian of F is given by

$$dF_{(t,x)} = \begin{pmatrix} 1 & 0 \\ \partial_t F_{\mathbb{R}^{n+1}}(t,x) & dF^t_x \end{pmatrix}.$$

The claim then is a corollary of Lemma 5.3.3. ∎

Definition 5.3.26. Such an extension will be called *cylinder-preserving extension* of f.

Definition 5.3.27. Let $f = \left(\mathrm{id}_{[-1,1]}, f_{S^n}\right) : \mathcal{Z} \to \mathcal{Z}$ be a map admissibly close to the identity. Let $U(\mathbb{1})$ be the corresponding very convex neighbourhood and consider $F : U(\mathcal{Z}) \to \mathbb{R} \times \mathbb{R}^{n+1}$ to be a *cylinder-preserving extension* of f. We then define

$$\begin{array}{cccc} \tilde{G}: & U(\mathcal{Z}) \times \mathfrak{U}(0) & \to & \mathbb{R}^{n+1} \\ & (t,x,g) & \mapsto & F_{\mathbb{R}^{n+1}}(t,x) - \exp(g)\cdot x \end{array}$$

and

$$\tilde{M} := \tilde{G}^{-1}(0),$$

where $\mathfrak{U}(0) = \exp^{-1}(U(\mathbb{1}))$

Lemma 5.3.28. *With the assumptions of the last definition $\tilde{M} \subset U(\mathcal{Z}) \times \underline{\mathfrak{so}}(n+1)$ is an embedded submanifold with boundary and* $\dim(\tilde{M}) = \frac{n(n+1)}{2} + 1$. *Moreover, $\tilde{M} \subset \mathcal{Z} \times \mathfrak{U}(0)$.*

Proof: The proof is an analogue to that of Lemma 5.3.8. We observe that $\tilde{G}(t,x,g) = 0$ implies $\|F_{\mathbb{R}^{n+1}}(t,x)\| = \|x\|$. By property (iib) of cylinder-preserving extensions this is equivalent to requiring $(t,x) \in \mathcal{Z}$ and hence $\tilde{G} \subset \mathcal{Z} \times \mathfrak{U}(0)$. It suffices to show that $d\tilde{G}_p$ has full rank for all $p = (t,x,g) \in \mathcal{Z} \times \mathfrak{U}(0)$. The differential $d\tilde{G}_{(t,x,g)} : \mathbb{R} \times \mathbb{R}^{n+1} \times T_g\underline{\mathfrak{so}}(n+1) \to \mathbb{R}^{n+1}$ is given by

$$\begin{aligned} d\tilde{G}_{(t,x,g)}(v_t, v_x, v_g) &= \left.\frac{d}{ds}\right|_{s=0} \tilde{G}(t+sv_t, x+sv_x, g+sv_g) \\ &= (dF_{\mathbb{R}^{n+1}})_{(t,x)}(v_t, v_x) - (d\exp)_g(v_g) \cdot x - \exp(g) \cdot v_x. \end{aligned}$$

Restricting to $(v_t, v_x) = (0,0)$ and using (1.88) gives

$$d\tilde{G}_{(t,x,g)}(0,0,T_g\underline{\mathfrak{so}}(n+1)) = T_{\exp(g)\cdot x}S^n = T_{f_{S^n}(t,x)}S^n$$

for all $(t,x) \in \mathcal{Z}$. Now choosing $(v_t, v_x, v_g) = (0,x,0)$ and using that f is admissibly close to the identity gives for each $(t,x) \in \mathcal{Z}$

$$d\tilde{G}_{(t,x,g)}(0,x,0) = dF^t_x(x) - \exp(g) \cdot x = f_{S^n}(t,x) \neq 0$$

On the left-hand side, $f_{S^n}(t,x)$ interpreted as a vector in the tangent space is transversal to the sphere. Hence $f_{S^n}(t,x)$ and $T_{f_{S^n}(t,x)}S^n$ span the full $\mathbb{R}^{n+1}$ and we conclude that $d\tilde{G}_{(t,x,g)}$ is surjective for all $(t,x,g) \in \mathcal{Z} \times \mathfrak{U}(0)$. By the regular value theorem $\tilde{G}^{-1}(0)$ is a $\frac{n(n+1)}{2}+1$ dimensional submanifold of $U(\mathcal{Z}) \times \mathfrak{U}(0)$ if it is not empty. Non-emptiness is provided by the requirement that f is admissibly close to the identity and so for each $(t,x) \in \mathcal{Z}$ there is at least one $g \in \mathfrak{U}(0)$ such that $\tilde{G}(t,x,g) = 0$. ■

Again we would like $d\pi_{\mathcal{Z}}$ to have full rank.

Lemma 5.3.29. *The projection map $\pi_{\mathcal{Z}} : \tilde{M} \ni (t,x,g) \mapsto (t,x) \in \mathcal{Z}$ is a surjective submersion.*

Proof: Surjectivity of $\pi_{\mathcal{Z}}$ is due to the fact that f is admissibly close to the identity. To show surjectivity of its differential $d\pi_{\mathcal{Z}}$, we calculate its kernel. On the one hand v_t and v_g will have to vanish for $(v_t, v_x, v_g) \in \ker d\pi_{\mathcal{Z}} \subset T_{(t,x,g)}\tilde{M}$. On the other hand $(0,0,v_g)$ will have to be an element of the kernel of $d\tilde{G}_{(t,x,g)}$, which is true if and only if $d\exp_g(v_g) \in T_{exp(g)}SO(n+1) = exp(g) \cdot \underline{\mathfrak{so}}(n+1)$ annihilates x. Hence we find

$$(v_t, v_x, v_g) \in \ker\left(d\pi_{\mathcal{Z}\,(t,x,g)}\right) \iff \begin{cases} (v_t, v_x) &= (0,0) \\ v_g &\in d\exp_g^{-1}\left(\exp(g) \cdot \underline{\mathfrak{stab}}(x)\right) \end{cases}$$

and $\dim\left(\ker \pi_{\mathcal{Z}\,(t,x,g)}\right) = \dim\left(\underline{\mathfrak{so}}(n)\right)$. The dimension of the image of $d\pi_{\mathcal{Z}}$ then is $\dim\left(\mathrm{im}(\pi_{\mathcal{Z}})_{(t,x,g)}\right) = \dim(\tilde{M}) - \dim\left(\underline{\mathfrak{so}}(n)\right) = n+1 = \dim(\mathcal{Z})$, which makes $d\pi_{\mathcal{Z}}$ a surjective map. ■

We use the previous lemma to find the following local result.

Lemma 5.3.30. *For each $(t,x) \in \mathcal{Z}$ there is a neighbourhood $U(t,x) \subset \mathcal{Z}$ and a map $g : U(t,x) \to \underline{\mathfrak{so}}(n+1)$ such that*

(i) $(s,y,g(s,y)) \in \tilde{M}$ *for all* $(s,y) \in U(t,x)$ *and*

(ii) g *is* C^m*-smooth.*

Proof: We will only sketch the proof here since it essentially agrees with that of Lemma 5.3.10. Let $(t,x,g) \in \tilde{M}$ be an arbitrary point. Since $\mathcal{Z} \times \underline{\mathfrak{so}}(n+1)$ is a manifold with boundary and $\tilde{M}$ is a submanifold of dimension $\frac{n(n+1)}{2}+1 =: n+1+d$, there is a C^m-chart $\varphi : U \subset \mathcal{Z} \times \underline{\mathfrak{so}}(n+1) \to \mathbb{R}^+_0 \times \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^n$ for a neighbourhood of (t,x,g) such that $\varphi(t,x,g) = 0$ and $\varphi(s,y,h) \in \mathbb{R}^+_0 \times \mathbb{R}^n \times \mathbb{R}^d \times \{0\}$ for all $(s,y,h) \in \tilde{M} \cap U$. Only the last $\mathbb{R}^n$-part in the decomposition refers to the special property of $\tilde{M}$ of being a submanifold. The remaining decomposition is arbitrary at the moment and will be justified later. In case where $-1 < t < 1$, $\mathbb{R}^+_0$ may be replaced by $\mathbb{R}$. It is a reference to the boundary of the interval $[-1,1]$. By Lemma 5.3.29 the composition

$\pi_{\mathcal{Z}} \circ \varphi^{-1} : \mathbb{R}^n \times \mathbb{R}_0^+ \times \mathbb{R}^d \times \{0\} \to \mathcal{Z}$ is a surjective submersion. Hence we may assume φ to be such that $\ker\left(\left(d\pi_{\mathcal{Z}} \circ \varphi^{-1}\right)_{\varphi(t,x,g)}\right) = \{0\} \times \{0\} \times \mathbb{R}^d \times \{0\}$. This may be achieved by a rotation in $\mathbb{R}^{n+1+d}$. The restriction $\tilde{\vartheta} := \pi_{\mathcal{Z}} \circ \varphi^{-1}|_{\mathbb{R}_0^+ \times \mathbb{R}^n \times \{0\} \times \{0\}}$ then is a diffeomorphism for some neighbourhood $U_0 \subset \mathbb{R}_0^+ \times \mathbb{R}^n \times \{0\} \times \{0\}$ of the origin. Altogether the previous sketch motivates the existence of a well defined map g in the following diagram for a neighbourhood of $(t,x) \in \mathcal{Z}$.

$$\begin{tikzcd}
\mathbb{R}_0^+ \times \mathbb{R}^n \times \{0\} \times \{0\} & \mathcal{Z} \times \underline{\mathfrak{so}}(n+1) & \underline{\mathfrak{so}}(n+1) \\
\cup & \cup & \\
U_0 \arrow[r, leftrightarrow, "\varphi^{-1}"'] \arrow[rr, leftrightarrow, bend right, "\tilde{\vartheta}"'] & \varphi^{-1}(U_0) \arrow[r, dashed, leftrightarrow, "\pi_{\mathcal{Z}}"'] \arrow[ur, "\pi_{\underline{\mathfrak{so}}(n+1)}"] & \mathcal{Z} \arrow[uu, "g"']
\end{tikzcd}$$

The map is defined by

$$g : \begin{array}{ccc} \tilde{\vartheta}(U_0) & \to & \underline{\mathfrak{so}}(n+1) \\ (s,y) & \mapsto & \pi_{\underline{\mathfrak{so}}(n+1)} \circ \varphi^{-1} \circ \tilde{\vartheta}^{-1}(s,y). \end{array}$$

Again the idea of the proof was to show that the dashed line indeed represents a bijective map. ∎

Making this a global result gives the following proposition

Proposition 5.3.31. *Let $f : \mathcal{Z} \to \mathcal{Z}$ be a C^m-smooth map admissibly close to the identity, $\tilde{M}$ the corresponding manifold defined above. Then there is a C^m-smooth map $g : \mathcal{Z} \to \underline{\mathfrak{so}}(n+1)$ such that*

$$(p, g(p)) \in \tilde{M} \tag{5.19}$$

for all $p = (t,x) \in \mathcal{Z}$. Consequently f can be written as

$$f(t,x) = (t, \exp(g(t,x)) \cdot x).$$

Proof: The reasoning of the proof coincides with that done in the last section. So we will only outline it here. Let $\mathfrak{U}(0) := \exp(U(\mathbb{1}))$ be the preimage of the very convex neighbourhood $U(\mathbb{1})$. Then for each $(t,x) \in \mathcal{Z}$ there is at least one $h^{t,x} \in \mathfrak{U}(0)$ such that $f(t,x) = (t, f_{S^n}(t,x)) = (t, \exp(h^{t,x}) \cdot x)$. We define a fibration of $\tilde{M}$ by

$$\{(t,x)\} \times \tilde{F}_{(t,x)} := \pi_{\mathcal{Z}}^{-1}((t,x)).$$

More explicitly we can write $\tilde{F}_{(t,x)} = \{h \in \mathfrak{U}(0) \parallel \exp(h) \cdot x = \exp(h^{t,x}) \cdot x\}$. By applying Lemma 5.3.11 we find the $\tilde{F}_{(t,x)}$ to be connected and it allows us to write it as

$$\tilde{F}_{(t,x)} = \exp^{-1}\left(\exp\left(h^{t,x}\right) \cdot \mathrm{stab}(x) \cap U(\mathbb{1})\right).$$

The next step is to construct a global C^m-smooth section of that fibration. Local sections are provided by Lemma 5.3.30. By further shrinking the neighbourhood, we find for each $p \in \mathcal{Z}$ an open ball $B_p \subset \mathcal{Z}$ and a C^m-map $g_p : B_p \to \mathfrak{U}(0)$ such that $(q, g_p(q)) \in \tilde{M}$ for all $q \in B_p$. By open ball, we mean the restriction of open balls in $\mathbb{R}^{n+2}$ to the cylinder $\mathcal{Z}$. Using compactness of $\mathcal{Z}$ we can choose a finite subcover $\mathcal{Z} = \bigcup_i B_i$ and local sections $g_i : B_i \to \mathfrak{U}(0)$. We observe that the double partition of unity provided by Lemma 5.3.17 and Corollary 5.3.18 and the gluing Lemma 5.3.19 not works only for maps on the sphere but also for maps on the compact cylinder $[-1,1] \times S^n$, since the only property needed in the proof is compactness. The remaining proof then can be carried out, using the induction method introduced in the proof of Proposition 5.3.20 ∎

A minor modification is required if $f : \mathcal{Z} \to \mathcal{Z}$ is a $C^{m>1}$-map only away from $t = 0$, but a C^{m-1}-map else. In that case Proposition 5.3.31 would provide only a C^{m-1}-map $g : \mathcal{Z} \to \underline{\mathfrak{so}}(n+1)$. To get a C^m-map away from $t = 0$ we have to require that all g_i have this property. Lemma 5.3.30 can be modified such that it supports that smoothness. Smoothness of the local sections g_i is inherited from the charts $\varphi : U \to \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^n$. For $t \neq 0$ we can choose a neighbourhood $U(t,x)$ of (t,x) such that $f : U(t,x) \to \mathcal{Z}$ is C^m-smooth and hence $\tilde{M}$ locally admits a C^m-chart for a neighbourhood of $(t,x,g) \in \tilde{M}$. It remains to consider points $(0,x,g) \in \tilde{M} \subset \mathcal{Z} \times \mathfrak{U}(0)$ and to construct a chart for a neighbourhood of them that is C^{m-1} for $t = 0$ and C^m else. The differential of $\tilde{G} : U(\mathcal{Z}) \times \mathfrak{U}(0) \to \mathbb{R}^{n+1}$ has full rank in $(0,x,g)$. The implicit function theorem then provides a map $\psi : U' \to U''$ such that

$$
\begin{aligned}
(U' \oplus U'') \cap \tilde{M} = \; &\{x \oplus \psi(x) \mid x \in U'\} \\
U' \subset \; &\ker\left(d\tilde{G}_{(0,x,g)}\right) \simeq \mathbb{R}^{\frac{n(n+1)}{2}+1} \\
U'' \subset \; &\ker\left(d\tilde{G}_{(0,x,g)}\right)^{\perp} \simeq \mathbb{R}^{n+1}.
\end{aligned}
$$

In particular $\tilde{G}(y \oplus \psi(y)) = 0$ for all $y \in U' \oplus U''$ and the differential of ψ is given by $d\psi_y = -d''\tilde{G}^{-1}_{y\oplus\psi(y)} \circ d'\tilde{G}_{y\oplus\psi(y)}$. Consequently the map $d\psi : y \mapsto d\psi_y$ inherits differentiability of $\tilde{G}$, which is of class C^{m-1} as long as $y \oplus \psi(y) \neq (0,\cdot)$. Finally, we can define the chart $\phi^{-1} : U' \simeq \tilde{M}$ by $\phi^{-1}(x) := x \oplus \psi(x)$.

Summarising the last paragraphs gives a modification of Proposition 5.3.31.

Proposition 5.3.32. *Let $f : \mathcal{Z} \to \mathcal{Z}$ be a C^{m-1}-smooth map admissibly close to the identity, which is C^m-smooth for point $(t,x) \in \mathcal{Z}$ with $t \neq 0$. Let $\tilde{M}$ be the corresponding manifold defined above. Then there is a C^{m-1}-smooth map $g : \mathcal{Z} \to \underline{\mathfrak{so}}(n+1)$, which is C^m-smooth away from $t = 0$ such that*

$$(p, g(p)) \in \tilde{M} \tag{5.20}$$

for all $p = (t,x) \in \mathcal{Z}$. Consequently f can be written as

$$f(t,x) = (t, \exp(g(t,x)) \cdot x).$$

5.3.3. Extension of Special Cone Diffeomorphisms

This section considers the extension of special diffeomorphism, given on a cone. The extension will later correspond to a coordinate transformation. The section is organised as follows. First we will specify the bijective map on the cone in $\mathbb{R}^n$ that we would like to expand to a map on $\mathbb{R}^n$. The second step is to blow up the vertex of the cone and to define a sufficiently smooth bijective map on the cylinder $\mathcal{Z} = [-1,1] \times S^{n-1}$, which emerges from the blow up. Using the results of the last section this can be lifted to a sufficiently smooth map on the cylinder with values in the Lie algebra $\underline{\mathfrak{so}}(n-1)$. This map can be extend to a map defined on $[-1,1] \times \mathbb{R}^{n-1}$ with values in the Lie algebra. The map will then be used to define an extension of the original bijection on the cone. Its smoothness properties and its bijectivity are topic of the last part of this section.

Definition 5.3.33. Let $\zeta : \mathbb{R}^n \supset U \to \mathbb{R}^n$ be a $C^{m\geq 2}$-diffeomorphism onto its image. It will be called *cone-preserving diffeomorphism close to the identity* if it has the following properties:

(i) $\zeta(x) = x + \mathrm{o}(\|x\|_n)$

(ii) $\zeta(y) \in \mathfrak{C} \Leftrightarrow y \in \mathfrak{C} \cap U$ where $\mathfrak{C} = \left\{y \in \mathbb{R}^n \mid \|y\|_{1,n-1} = 0\right\}$ is the Minkowski cone in $\mathbb{R}^{1,n-1}$.

So for the rest of this section let ζ be a cone-preserving diffeomorphism close to the identity. For the moment consider the rescaled map

$$
\begin{aligned}
f : \quad \mathbb{R}^n \quad &\to \quad \mathbb{R}^n \\
y \quad &\mapsto \begin{cases} \frac{\|y\|_n}{\|\zeta(y)\|_n}\zeta(y) & y \neq 0 \\ 0 & \text{else.} \end{cases}
\end{aligned}
\tag{5.21}
$$

It is of class C^1 in a neighbourhood of the origin[6]. Moreover it is a cone-preserving diffeomorphism close to the identity. An important property of f is that it preserves spheres of type $\left(\{r\} \times S_r^{n-2}\right)$ on the cone $\mathfrak{C} = \{y \in \mathbb{R}^n | \|y\|_{1,n-1} = 0\}$. Aim of this section is to extend the restriction $f|_{\mathfrak{C}}$ to a map on a neighbourhood of the origin such that the extension more generally preserves spheres of type $\left(\{t\} \times S_r^{n-2}\right)$ with $t \neq r$. The map that we are going to construct will have some additional important properties that will be summarised at the end of this section.

Initially consider the map

$$\beta = \left(\beta^0, \boldsymbol{\beta}\right) : (\mathbb{R} \setminus \{0\}) \times \mathbb{R}^{n-1} \ni (t,x) \mapsto t\,(1,x) \in (\mathbb{R} \setminus \{0\}) \times \mathbb{R}^{n-1}.$$

It maps cylinders in $\mathbb{R}^n$ to cones and preserves the first component (see figure 9 for a visualisation of β). In particular points (t,x) with $\|x\|_{n-1} = c = \text{const.}$ have images with $\frac{\|\boldsymbol{\beta}(t,x)\|_{n-1}}{\beta^0(t,x)} = c$. Its inverse $\beta^{-1} : (\mathbb{R} \setminus \{0\}) \times \mathbb{R}^{n-1} \to (\mathbb{R} \setminus \{0\}) \times \mathbb{R}^{n-1}$ is provided by $\beta^{-1}(t,x) = \left(t, \frac{x}{t}\right)$. Now for $\xi = \left(\xi^0, \boldsymbol{\xi}\right) \in C^m\left(U \subset \mathbb{R}^n, \mathbb{R} \times \mathbb{R}^{n-1}\right)$ consider $\tilde{U} = \beta^{-1}\left(U \setminus \left(\left(\xi^0\right)^{-1}(0) \cup \{0\} \times \mathbb{R}^{n-1}\right)\right)$ and let ξ^β be defined by

$$\begin{array}{rccc} \xi^\beta : & (\mathbb{R} \setminus \{0\}) \times \mathbb{R}^{n-1} \supset \tilde{U} & \to & \mathbb{R}^n \\ & y = (t,x) & \mapsto & \left(t, \pi^{\mathbb{R}^{n-1}} \circ \beta^{-1} \circ \xi \circ \beta(y)\right) \end{array}$$

where the projection is with respect to the last $n-1$ components of $\beta^{-1} \circ \xi \circ \beta$. More explicitly (t,x) is mapped to $\left(t, \frac{\boldsymbol{\xi}(t(1,x))}{\xi^0(t(1,x))}\right)$ and one can see that for the transformation of two different maps one has the equality $\xi^\beta = \tilde{\xi}^\beta$ if $\xi = \lambda\tilde{\xi}$ for a nowhere vanishing function λ. We also have the following lemma for the transformation of diffeomorphisms close to the identity.

Lemma 5.3.34. *Let $\xi \in C^{m>1}\left(U \subset \mathbb{R}^n, \mathbb{R}^n\right)$ be a diffeomorphism on a star shaped neighbourhood U of 0 with $\xi(x) = x + o(\|x\|)$. Then ξ^β as defined above is a C^m map and admits a C^{m-1}-smooth extension to points at the hyperplane $\{0\} \times \mathbb{R}^{n-1}$, which is provided by*

$$\begin{array}{rccc} \xi^\beta : & \tilde{U} \cup \left(\{0\} \times \mathbb{R}^{n-1}\right) & \to & \mathbb{R}^n \\ & y = (t,x) & \mapsto & \begin{cases} \left(t, \pi^{\mathbb{R}^{n-1}} \circ \beta^{-1} \circ \xi \circ \beta(y)\right) & t \neq 0 \\ y & \text{else} \end{cases} \end{array}$$

The extension will also be denoted by ξ^β. Moreover the extension is a local diffeomorphism for points at the hyperplane $\{0\} \times \mathbb{R}^{n-1}$.

Proof: The claim is a direct consequence of Taylor's theorem with remainder (e.g. [Tu11]) as it allows one to write the map ξ as $\xi(x) = J(x) \cdot x$, where $J : U \to \mathrm{Mat}(n, \mathbb{R})$ is a matrix valued C^{m-1}-map[7] on U with $J(0) = d\xi_0 = \mathbb{1}$. Hence $\xi^\beta(t,x) = \left(t, \frac{\boldsymbol{\xi}(t(1,x))}{\xi^0(t(1,x))}\right) = \left(t, \frac{\pi^{\mathbb{R}^{n-1}}(J(t(1,x))\cdot(1,x))}{\pi^0(J(t(1,x))\cdot(1,x))}\right)$. Numerator and denominator in the fraction are C^{m-1}-maps with the denominator being 1 at the origin. Hence ξ^β at least locally is a composition of C^{m-1}-maps which provides the first part of the claim. The second part is an immediate consequence, as the fraction is well defined for all $(t,x) \in \tilde{U} \cup \left(\{0\} \times \mathbb{R}^{n-1}\right)$. The differential of the extended ξ^β then has full rank along the hyperplane $\{0\} \times \mathbb{R}^{n-1}$ which makes it a local diffeomorphism along the hyperplane. ■

The analogue definition for a reverse transformation then gives

$$\begin{aligned} \left(\xi^\beta\right)^{\beta^{-1}}(t,x) &= \beta \circ \xi^\beta \circ \beta^{-1}(t,x) \\ &= \beta\left(t, \frac{\boldsymbol{\xi}(t,x)}{\xi^0(t,x)}\right) \\ &= \frac{t}{\xi^0(t,x)}\xi(t,x) \end{aligned}$$

6 The map f is defined only for the purpose of motivation of further constructions. So no smoothness issues will be proven explicitly. On the other hand the proofs are kind of hidden in the following work.

7 The map J is defined by $J(y) = \int_0^1 d\xi_{sy} ds$, where $d\xi_{sy}$ is the Jacobian of ξ at sy.

for $\xi^0(t,x) \neq 0$, which along the cone $\mathfrak{C}$ coincides with $\frac{\|y\|_n}{\|\xi(y)\|_n}\xi(y)$ if ξ preserves the cone $\mathfrak{C}$.

Now consider the cone-preserving diffeomorphism ζ we have started with and let $f^{\mathcal{Z}} := \zeta^\beta\big|_{\mathcal{Z}_\epsilon} : \mathcal{Z}_\epsilon \to \mathcal{Z}_\epsilon$ be the restriction of ζ^β to the cylinder $\mathcal{Z}_\epsilon = (-\epsilon,\epsilon) \times S^{n-2}$. The aim of the remaining part of the section will be to extend $f^{\mathcal{Z}}$ to a map on $(-\epsilon,\epsilon) \times \mathbb{R}^{n-1}$ such that $\left(f^{\mathcal{Z}}\right)^{\beta^{-1}}$ extends $f|_{\mathfrak{C}}$ to a map on $(-\epsilon,\epsilon) \times \mathbb{R}^{n-1}$ which preserves spheres of type $\{t\} \times S^{n-2}_r$. As a matter of fact $\left(f^{\mathcal{Z}}\right)^{\beta^{-1}}(y) = \frac{\|y\|_n}{\|\zeta(y)\|_n}\zeta(y) = f(y)$ is granted by the previous considerations on the map ξ. As $f^{\mathcal{Z}}$ is the restriction of a local C^{m-1}-diffeomorphism in a neighbourhood of the hyperplane $\{0\} \times \mathbb{R}^{n-1}$ by Lemma 5.3.34, it is a local diffeomorphism if ϵ is has been chosen small enough. Moreover it then is bijective by Lemma 5.3.2, as it preserves spheres and hence is a C^{m-1}-diffeomorphism on $\mathcal{Z}_\epsilon$. Lemma 5.3.34 also provides that $f^{\mathcal{Z}}$ is of class C^m for $(t,x) \in \mathcal{Z}_\epsilon$ with $t \neq 0$. We now may assume $\epsilon > 1$. If not, we may continue the following calculations with a scaled $f^{\mathcal{Z}}$, i.e. a map defined by $f^{\mathcal{Z},\delta}(y) := \frac{1}{\delta} f^{\mathcal{Z}}(\delta y)$ for δ being sufficiently small. So far we have shown that $f^{\mathcal{Z}}$ is a diffeomorphism on the compact cylinder $\mathcal{Z} : [-1,1] \times S^{n-2}$ which is admissibly close to the identity and fulfils the requirements of Proposition 5.3.32. Hence, there is a C^{m-1}-map $g : \mathcal{Z} \to \underline{\mathfrak{so}}(n-1)$, which is of class C^m away from $\{0\} \times S^{n-2}$ and it holds $f^{\mathcal{Z}}(t,x) = (t, \exp(g(t,x)) \cdot x)$.

Remark 5.3.35. Along the sphere $\{0\} \times S^{n-2}$ we have $f^{\mathcal{Z}}(0,x) = (0,x)$. Lemma 5.3.11 then implies $g(0,x) \in \underline{\mathfrak{stab}}(x)$.

We will now define an extension $F^{\mathcal{Z}} : [-1,1] \times \mathbb{R}^{n-1} \to \mathbb{R}^n$ which coincides with $f^{\mathcal{Z}}$ along the cylinder $\mathcal{Z}$. For that assume $\kappa : \mathbb{R}^{n-1} \to [0,1]$ to be a C^∞-function with the following properties

(i) κ is constant along spheres, i.e. $\|x\|_{n-1} = \|y\|_{n-1} \Rightarrow \kappa(x) = \kappa(y)$.

(ii) If $\|x\|_{n-1} \in \left(\frac{3}{4}, \frac{5}{4}\right)$ then $\kappa(x) = 1$.

(iii) If $\|x\| < \frac{1}{2}$ or $\frac{3}{2} < \|x\|$ then $\kappa(x) = 0$.

Remark 5.3.36. Roughly speaking in cylindrical coordinates κ is a bump function that depends only on the radius. The last three requirements also imply two more properties that are important for later considerations

(iv) The support of κ is subset of the compact set $\left\{x \in \mathbb{R}^{n-1} \mid \|x\| \in \left[\frac{1}{2}, \frac{3}{2}\right]\right\}$. In particular its derivatives are compactly supported. Hence $\|d\kappa_x\|$ is bounded, i.e. $\|d\kappa_x(V)\| < K < \infty$ for some $K \in \mathbb{R}$ and for all $V \in S^{n-2}$.

(v) $\kappa(x)$ and $d\kappa_x$ vanish outside $\left[\frac{1}{2}, \frac{3}{2}\right]$. This implies $\|x\kappa(x)\| < \frac{3}{2}$ and $\|x \cdot d\kappa_x(x)\| \leq \frac{9K}{4}$.

We now define the extension of $f^{\mathcal{Z}}$ as follows

$$\begin{array}{rccc} F^{\mathcal{Z}} = \left(F_1^{\mathcal{Z}}, F_2^{\mathcal{Z}}\right): & [-1,1] \times \mathbb{R}^{n-1} & \to & [-1,1] \times \mathbb{R}^{n-1} \\ & (t,x) & \mapsto & \begin{cases} \left(t, \exp\left(\kappa(x) g\left(t, \frac{x}{\|x\|}\right)\right) \cdot x\right) & x \neq 0 \\ (t,x) & \text{else} \end{cases} \end{array}.$$

The map is C^{m-1} everywhere and C^m for $t \neq 0$ since $\kappa(x) = 0$ for a neighbourhood of 0. In the following considerations we will not care about the line $[-1,1] \times \{0\}$, where x vanishes, since $F^{\mathcal{Z}}$ is the identity in a neighbourhood of that line and hence well defined.
Now we are able to define the extension of $f|_{\mathfrak{C}} : \mathfrak{C} \to \mathfrak{C}$ by $F =: \left(F^{\mathcal{Z}}\right)^{\beta^{-1}}$ and more explicitly by

$$\begin{array}{rccc} F = \left(F^0, \mathbf{F}\right): & [-1,1] \times \mathbb{R}^{n-1} & \to & [-1,1] \times \mathbb{R}^{n-1} \\ & (t,x) & \mapsto & \begin{cases} \left(t, tF_2^{\mathcal{Z}}\left(t, \frac{x}{t}\right)\right) & t \neq 0 \\ (0,x) & \text{else} \end{cases} \end{array}.$$

We will now show that F is a C^m-smooth map, except in $(t,x) = 0$, where it is C^1. In the end of this section we will point out the important properties of this map.

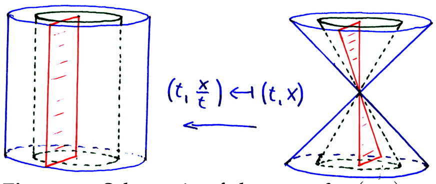

Figure 9.: Schematic of the map $\beta : (t,x) \mapsto (t,tx)$, which is used to transform cones into cylinders and vice versa.

First we rewrite F away from $t = 0$ using the map $g : \mathcal{Z} \to \underline{\mathfrak{so}}(n-1)$. Then we have

$$F(t,x) = \left(t, \exp\left(\kappa\left(\frac{x}{t}\right) g\left(t, \frac{x}{\|x\|}\right)\right) \cdot x\right) \tag{5.22}$$

and C^m-smoothness is clear for $t \neq 0$. So we have to check the limits $(t,x) \to (0,x_0)$ for F and its derivatives. We have $\kappa \equiv 0$ at the set $\left\{(t,x) \in [-1,1] \times \mathbb{R}^{n-1} \,|\, t \neq 0, \left\|\frac{x}{t}\right\| > 3/2\right\}$. This set describes the exterior of a cone with vertex at the origin. In particular F is the identity there and as long as $x_0 \neq 0$, extending F to points $(0,x_0)$ by the identity gives a C^∞-smooth map there. It remains to check C^1 smoothness in $(t,x) = 0$.

We will show the claim for the last $n-1$ components of F, as it is obvious for the first one. The norm of $\mathbf{F(t,x)}$ independently of t is $\left\|\exp\left(\kappa\left(\frac{x}{t}\right) g\left(t, \frac{x}{\|x\|}\right)\right) \cdot x\right\|_{n-1} = \|x\|_{n-1}$, which immediately gives continuity in $(t,x) = 0$.

For C^1-smoothness we derive the differential of $\mathbf{F}$ away from $t = 0$ and then determine its limit as (t,x) goes to zero.

$$\begin{aligned}(d\mathbf{F})_{(t,x)}(V_1,V_2) &= V_1 F_2^{\mathcal{Z}}\left(t, \frac{x}{t}\right) + t\left(dF_2^{\mathcal{Z}}\right)_{\left(t,\frac{x}{t}\right)}\left(V_1, \frac{V_2}{t} - \frac{V_1}{t}\frac{x}{t}\right) \\ &= V_1\left(F_2^{\mathcal{Z}}\left(t, \frac{x}{t}\right) + \left(dF_2^{\mathcal{Z}}\right)_{\left(t,\frac{x}{t}\right)}\left(t, -\frac{x}{t}\right)\right) + \left(dF_2^{\mathcal{Z}}\right)_{\left(t,\frac{x}{t}\right)}(0,V_2).\end{aligned}$$

First we consider $V_1 = 0$. Then $(d\mathbf{F})_{(t,x)}(0,V_2) = \left(dF_2^{\mathcal{Z}}\right)_{\left(t,\frac{x}{t}\right)}(0,V_2)$. It suffices to show that in uniform norm $\lim_{t\to 0}\left\|\left(dF_2^{\mathcal{Z}}\right)_{(t,\cdot)}(0,V_2) - V_2\right\|_{\mathbb{R}^{n-1}} = 0$. The map $F^{\mathcal{Z}}$ is of class C^{m-1} and coincides with the identity for (t,y) and $\|y\|_{n-1} > 3/2$. In particular $\left(dF_2^{\mathcal{Z}}\right)_{(t,y)}(0,V_2) = V_2$ for all $\|y\|_{n-1} > 3/2$ and hence $\left\|\left(dF_2^{\mathcal{Z}}\right)_{(t,\cdot)}(0,V_2) - V_2\right\|_{B^c_{3/2}(0)} = 0$, where $B^c_{3/2}(0)$ is the complement of the compact ball $B_{3/2}(0) \subset \mathbb{R}^{n-1}$ with radius $3/2$. On the other hand $\left(dF_2^{\mathcal{Z}}\right)_{(0,y)}(0,V_2) = V_2$ can be obtained by a direct calculation[8]. Hence on the compact ball $B_{3/2}(0) \subset \mathbb{R}^{n-1}$ the limit $\lim_{t\to 0}\left\|\left(dF_2^{\mathcal{Z}}\right)_{(t,\cdot)}(0,V_2) - V_2\right\|_{B_{3/2}(0)}$ exists and is zero. Combining this consequence with the first consideration on the complement of $B_{3/2}$ then provides uniform convergence all over $\mathbb{R}^{n-1}$.

Now we take care of the case where $(V_1,V_2) = (1,0)$. By using Equation (5.22) we get away from the origin

$$\begin{aligned}(d\mathbf{F})_{(t,x)}(1,0) &= \left.\frac{d}{ds}\right|_{s=0} \exp\left(\kappa\left(\frac{x}{t+s}\right) g\left(t+s, \frac{x}{\|x\|}\right)\right) \cdot x \\ &= d\exp_{\kappa\left(\frac{x}{t}\right)g\left(t,\frac{x}{\|x\|}\right)}\left(dg_{\left(t,\frac{x}{\|x\|}\right)}(1,0)\right) \cdot \kappa\left(\frac{x}{t}\right)\frac{x}{t}t \\ &\quad - d\exp_{\kappa\left(\frac{x}{t}\right)g\left(t,\frac{x}{\|x\|}\right)}\left(g\left(t, \frac{x}{\|x\|}\right)\right) \cdot d\kappa_{\frac{x}{t}}\left(\frac{x}{t}\right)\frac{x}{t}.\end{aligned} \tag{5.23}$$

By Remark 5.3.36(v), $\kappa\left(\frac{x}{t}\right)\frac{x}{t}$ is bounded and hence the first term converges to $(1,0)$ in any limit where t is sent to zero. Also by Remark 5.3.36(v) the factor $d\kappa_{\frac{x}{t}}\left(\frac{x}{t}\right)\frac{x}{t}$ in the second term is bounded and we may assume

$$d\kappa_{\frac{x}{t}}\left(\frac{x}{t}\right)\frac{x}{t} =: \lambda(t,x) \cdot \frac{x}{\|x\|},$$

where $\lambda : [-1,0) \cup (0,1] \times \mathbb{R}^{n-1} \to \mathbb{R}$ is a bounded not necessarily extendible map, which vanishes if $x = 0$. Hence for convergence of the second term in (5.23) it suffices to show vanishing

8 For the direct calculation we first observe $\kappa(y)g\left(0, \frac{y}{\|y\|}\right) \in \underline{\mathfrak{stab}}\left(\frac{y}{\|y\|}\right)$. This then gives $F_2^{\mathcal{Z}}(0,y) = (0,y)$. Using that $F^{\mathcal{Z}}$ is of class C^1 then gives the claimed equation.

of $d\exp_{\kappa\left(\frac{x}{t}\right)g\left(t,\frac{x}{\|x\|}\right)}\left(g\left(t,\frac{x}{\|x\|}\right)\right)\cdot\frac{x}{\|x\|}$ in the limit $t\to 0$. For that we use that with $X\in\underline{\mathfrak{so}}(n-1)$ and $\alpha\in\mathbb{R}$ we also have $d\exp_{\alpha X}(X)=\exp(\alpha X)\cdot X$ and hence

$$d\exp_{\kappa\left(\frac{x}{t}\right)g\left(t,\frac{x}{\|x\|}\right)}\left(g\left(t,\frac{x}{\|x\|}\right)\right)\cdot\frac{x}{\|x\|}=\exp\left(\kappa\left(\frac{x}{t}\right)g\left(t,\frac{x}{\|x\|}\right)\right)\cdot g\left(t,\frac{x}{\|x\|}\right)\cdot\frac{x}{\|x\|}.$$

The first factor on the right-hand side clearly has operator norm 1. So it suffices to show that $(y\mapsto g(t,y)\cdot y)$ uniformly converges to zero in the limit $t\to 0$. This is provided, as by Remark 5.3.35 we have $g(0,y)\in\underline{\mathfrak{stab}}(y)$ for all $y\in S^{n-2}$ and hence $\|g(t,y)y\|_{y\in S^{n-2}}\xrightarrow{t\to 0}0$ by compactness of S^{n-2} and continuity of g. Finally, we find $(d\mathbf{F})_{(t,x)}(1,0)\xrightarrow{(t,x)\to 0}0$ and hence $dF_{(t,x)}(1,0)\xrightarrow{(t,x)\to 0}(1,0)$.

Altogether we get

$$\lim_{(t,x)\to 0}dF_{(t,x)}=\mathrm{id}\,.$$

This coincides with the differential of F if derived using partial derivatives in 0. In particular $dF_0\left(0,V_2\right)=V_2$ since $F(0,x)=(0,x)$ and $dF_0(1,0)=(1,0)$ since $F(t,0)=(t,0)$ for all $t\in[-1,1]$ and $x\in\mathbb{R}^{n-1}$.

Now as we know F to be the identity in a neighbourhood of the punctured hyperplane $\{0\}\times\mathbb{R}^{n-1}\setminus\{0\}$ and to be a diffeomorphism in a neighbourhood of the origin, we can conclude that if ϵ is chosen small enough, then $F:(-\epsilon,\epsilon)\times\mathbb{R}^{n-1}\to(-\epsilon,\epsilon)\times\mathbb{R}^{n-1}$ is a diffeomorphism.

As announced in the beginning of this section, we will now point out the important properties of F.

Proposition 5.3.37. *Let $\zeta:\mathbb{R}^n\to\mathbb{R}^n$ be a cone-preserving $C^{m\geq 2}$-diffeomorphism that is close to the identity. Then there exists an $\epsilon>0$ and a cone-preserving C^1-diffeomorphism $F:(-\epsilon,\epsilon)\times\mathbb{R}^{n-1}\to\mathbb{R}^n$ close to the identity, which has the following properties. It*

(i) is C^m-smooth away from $(t,x)=0$.

(iia) preserves cylinders, i.e. $F\left((-\epsilon,\epsilon)\times S_r^{n-2}\right)=(-\epsilon,\epsilon)\times S_r^{n-2}$ for an arbitrary radius $r\in\mathbb{R}^+$.

(iib) preserves $\{t\}\times\mathbb{R}^{n-1}$-hypersurfaces, i.e. $F\left(\{t\}\times\mathbb{R}^{n-1}\right)=\{t\}\times\mathbb{R}^{n-1}$.

(iii) is a multiple of ζ along the cone $\mathfrak{C}=\left\{(t,x)\mid\|(t,x)\|_{1,n-1}^2=0\right\}$, i.e. $\zeta(y)=\frac{\|\zeta(y)\|}{\|y\|}F(y)$.

One may equivalently replace (iia) and (iib) by a point (ii) and instead observe that F preserves spheres of type $\{t\}\times S_r^{n-2}$ for $(t,r)\in(-\epsilon,\epsilon)\times\mathbb{R}^+$.

5.3.4. Coordinates Prescribing the Conformal Factor and Null Pregeodesics Originating at Vertices

The results of the last section will now be used to construct special coordinates for a neighbourhood of vertices of Σ. In those coordinates, null geodesics originating from the vertex are mapped to straight lines, while at the same time σ is a homogeneous polynomial of degree 2. In particular it has the same form that is provided by the Morse lemma for a neighbourhood of the vertex.

Consider $p\in\Sigma_d$ to be a vertex of Σ. Then by Proposition 5.1.1 there are a neighbourhood U of p and Morse coordinates $\varphi:U\to\mathbb{R}^n$ such that σ is of the form

$$s\sigma=-\left(\varphi^0\right)^2+\left(\varphi^1\right)^2+\cdots+\left(\varphi^{n-1}\right)^2$$

with $s=\pm 1$. Without loss of generality we assume the neighbourhood U to be a normal neighbourhood. By Lemma 5.1.3, the frame $\left\{\partial_i:=\frac{\partial}{\partial\varphi^i}\right\}$ is orthogonal at the vertex. The base vectors do not have to be of length 1 or -1. Nevertheless the frame provides a natural identification $I : \mathbb{R}^n \ni Y \mapsto \sum_i Y^i \partial_i T_p M$. We denote the geodesic coordinates that are defined by the inverse exponential map and I by $\tilde{\varphi} := I^{-1} \circ \exp^{-1} =: U \to \mathbb{R}^n$. In these new geodesic coordinates we have $\tilde{\partial}_i(p) = \partial_i(p)$ at the vertex, where $\tilde{\partial}_i := \frac{\partial}{\partial \tilde{\varphi}^i}$. The curves $\tilde{\gamma}(t) = \tilde{\varphi}^{-1}(t(1, \mathbf{e}))$ with unit vector $\mathbf{e} \in \mathbb{R}^{n-1}$ are null geodesics since $I(1, \mathbf{e})$ is a null vector. In particular the image of the cone $\mathfrak{C} = \left\{ x \in \tilde{\varphi}(U) \;\middle|\; (x^0)^2 = (x^1)^2 + \cdots + (x^{n-1})^2 \right\}$ under $\tilde{\varphi}^{-1}$ is the geodesic null cone $C_p(U)$ in p. By Proposition 5.1.12 it also coincides with $\Sigma \cap U$.

We point out two important facts. The first is that $\varphi(\Sigma \cap U)$ is a subset of the cone $\mathfrak{C}$. The second is that $\tilde{\varphi}(\Sigma \cap U)$ also is a subset of $\mathfrak{C}$. Let $\zeta := \tilde{\varphi} \circ \varphi^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ be the smooth coordinate transformation, which maps Morse to geodesic coordinates in p. Then by construction ζ is cone-preserving, $\zeta(0) = \tilde{\varphi} \circ \varphi^{-1}(0) = 0$ and for canonic base vectors $e_\mu \in \mathbb{R}^n$ we have $d\zeta_0(e_\mu) = d\tilde{\varphi}_p \circ \left(d\varphi^{-1}\right)_0 (e_\mu) = d\tilde{\varphi}_p\left((\partial_\mu)_p\right) = d\tilde{\varphi}_p\left((\tilde{\partial}_\mu)_p\right) = e_\mu$. In particular $d\zeta_0 = \mathrm{id}$ and so ζ is a cone-preserving diffeomorphism close to the identity. By Proposition 5.3.37, at least locally there exists a special C^1-smooth cone-preserving diffeomorphism $F = \left(F^0, \mathbf{F}\right) : \varphi(U) \supset \mathcal{U} \to \mathbb{R} \times \mathbb{R}^{n-1}$, close to the identity, which extends $\frac{\|y\|}{\|\zeta(y)\|}\zeta(y)$ away from the cone $\mathfrak{C} \cap \mathcal{U}$ in the suitable way described before. We now define new coordinates $\vartheta : U_p = \varphi^{-1}(\mathcal{U}) \to \mathbb{R}^n$ by

$$\vartheta := F \circ \varphi. \tag{5.24}$$

Now due to F fulfilling (i)-(iii) of Proposition 5.3.37, the new coordinates have the following properties.

(i) ϑ is smooth away from p and of class C^1 at p.

(ii) $\{\partial_{\vartheta^i}(p)\}$ is an orthogonal basis of T_pM.

(iii) Curves $\gamma : t \mapsto \vartheta^{-1}(t(1, \mathbf{e}))$ with $\|\mathbf{e}\|_{n-1} = 1$ are null pregeodesics in Σ originating in p.

(iv) σ in these coordinates is given by $s\sigma = -\left(\vartheta^0\right)^2 + \left(\vartheta^1\right)^2 + \cdots + \left(\vartheta^{n-1}\right)^2$.

The coordinate map ϑ inherits the smoothness of F, since φ is smooth. This gives the first point. The second point is a consequence of F being close to the identity, i.e. $dF_0 = \mathrm{id}$ and hence $\left(\partial_{\varphi^i}\right)_p = d(\vartheta^{-1} \circ \vartheta)_p\left(\left(\partial_{\varphi^i}\right)_p\right) = d\vartheta_0^{-1} \circ dF_0 \circ d\varphi_p\left(\left(\partial_{\varphi^i}\right)_p\right) = (\partial_{\vartheta^i})_p$. For the third point consider the curve $\gamma : t \mapsto \vartheta^{-1}(t(1, \mathbf{e}))$ and the geodesic coordinates $\tilde{\varphi}$, then

$$\begin{aligned}
\tilde{\varphi}(\gamma(t)) &= \zeta \circ \varphi(\gamma(t)) \\
&= \frac{\|\zeta \circ \varphi(\gamma(t))\|}{\|\varphi(\gamma(t))\|} F \circ \varphi(\gamma(t)) \\
&= \frac{\|\zeta \circ \varphi(\gamma(t))\|}{\|\varphi(\gamma(t))\|} \vartheta(\gamma(t)) \\
&= \frac{\|\zeta \circ \varphi(\gamma(t))\|}{\|\varphi(\gamma(t))\|} t(1, \mathbf{e}).
\end{aligned}$$

Hence γ has to be a null pregeodesic originating in $p \in M$. For the last point we use that F preserves cylinders and $\{t\} \times \mathbb{R}^{n-1}$-hypersurfaces (Proposition 5.3.37(ii)) and get

$$\begin{aligned}
-\left(\vartheta^0(y)\right)^2 + \left(\vartheta^1(y)\right)^2 + \cdots + \left(\vartheta^{n-1}(y)\right)^2 &= -\left|F^0 \circ \varphi(y)\right|^2 + \|\mathbf{F} \circ \varphi(y)\|_{n-1}^2 && (5.25) \\
&= -\left|\varphi^0(y)\right|^2 + \left\|\left(\varphi^1(y), \ldots, \varphi^{n-1}(y)\right)\right\|^2 && (5.26) \\
&= \sigma(y). && (5.27)
\end{aligned}$$

We will now summarise the substance of the last sections in a theorem.

Theorem 5.3.38. *Let $p \in \Sigma_d$ be a vertex of Σ, then for a neighbourhood U of p there exists special coordinates $\vartheta : U \to \mathbb{R}^n$ such that σ has the form*

$$s\sigma = -\left(\vartheta^0\right)^2 + \left(\vartheta^1\right)^2 + \cdots + \left(\vartheta^{n-1}\right)^2$$

with $s = \pm 1$. These coordinates have the smoothness of σ except in p where they are of class C^1. In addition a curve of the form

$$\gamma(t) = \vartheta^{-1}(tV)$$

is a null pregeodesic for all $(0 \neq V) \in \mathbb{R}^n$ with $\|V\|^2_{1,n-1} = 0$.

Coordinates Adapted to the Null Cone

The potential advantage of the coordinates in the last theorem is a reduction in the number of variables in the partial differential equation arising from the Cauchy problem defined by the almost Einstein equations. The defining function is prescribed in these coordinates and only the metric and its derivatives are considered to be unknowns to the system. Also the null direction along the cone is provided by straight lines in these coordinates.

The method of introducing special coordinates is used in general relativity to handle parts of the characteristic Cauchy problem for the vacuum field equations and to handle the conformal wave equations with initial data at a characteristic cone [Ren90, CBCMG11b, CP13]. We will partially review the method here and point out the modifications that have to be made to prescribe the conformal factor in such coordinates. A basic idea is to start with geodesic coordinates φ at the vertex p of the characteristic cone. If those coordinates are based on an orthogonal frame, this guarantees that the geodesic null cone in p is mapped to the Minkowskian null cone in $\mathbb{R}^n$. On the other hand, geodesics originating at the vertex are affinely parametrised in the sense that $\gamma(t) = \varphi^{-1}(tV)$ for some $V \in T_pM$. We will drop that last requirement in order to have the conformal factor fully prescribed in our coordinates

Let $p \in \Sigma_d$ be a vertex of Σ. By Theorem 5.3.38 there are a neighbourhood U of p and coordinates $\vartheta : U \to \mathbb{R}^n$ such that $\sigma = -\left(\vartheta^0\right)^2 + \left(\vartheta^1\right)^2 + \cdots + \left(\vartheta^{n-1}\right)^2$ and in addition null geodesics originating in p are mapped to straight lines, i.e. are of of the form $\gamma(t) = \vartheta^{-1}(\phi(t)V)$ with strictly increasing smooth function $\phi : [0,\epsilon) \to \mathbb{R}$ and a vector V with $\|V\|^2_{1,n-1} = 0$. In this way Σ locally is given by the equation for the Minkowskian null cone. One can now define coordinates $x : \mathcal{U} \supset U \to \mathbb{R} \times \mathbb{R}^+_0 \times \mathbb{R}^{n-2}$ for a subset of the local future-directed causal cone in p [CBCMG11b] by

$$\begin{aligned} x^0 &= r \circ \vartheta - \vartheta^0 \\ x^1 &= r \circ \vartheta \\ x^A &= \varphi^A\left(\frac{\left(\vartheta^1, \ldots, \vartheta^{n-1}\right)}{r \circ \vartheta}\right), \end{aligned}$$

where $(r \circ \vartheta)^2 := \left(\vartheta^1\right)^2 + \cdots + \left(\vartheta^{n-1}\right)^2$, $A \in \{2, \ldots, n-1\}$ and $\varphi = \left(\varphi^2, \ldots, \varphi^{n-1}\right) : \mathcal{S} \subset S^{n-2} \to \mathbb{R}^{n-2}$ are local spherical coordinates. The coordinates x are singular along the line $x^1 = 0$ and in particular at the vertex p. Null geodesics originating in p have $x^0 = 0$, $x^A = const.$ and are parametrised by x^1. In contrast to the construction in [CBCMG11b], where the latter coordinate component is an affine parameter, this is not the case here, as we did not start with geodesic coordinates. Nevertheless the coordinate derivative ∂_{x^1} is a null vector tangent to null geodesics originating in p. Moreover along the null cone $\mathcal{C}_p(\mathcal{U})$ it is perpendicular to the tangent vectors ∂_{x^A} for $A > 1$, since $\{x^1, \ldots, x^{n-1}\}$ parametrises $\mathcal{C}_p(\mathcal{U}) \supset \left(x^0\right)^{-1}(0)$. And hence along the cone one has $g_{11} = g_{1A} = 0$ for $A \in \{2, \ldots, n-1\}$. The cone itself and in particular Σ are characterised by the equation $x^0 = 0$. Using the notation of the latter paper, along the cone the metric then is of type

$$g_{\mu\nu} =|_{\mathcal{C}_p(\mathcal{U})} \begin{pmatrix} g_{00} & \nu_0 & \nu \\ \nu_0 & 0 & 0 \\ \nu^t & 0 & \tilde{g} \end{pmatrix},$$

where $\nu_0 = g_{01}$, $\nu : M \to \mathbb{R}^{n-1}$ is defined by $\nu_A := g_{0A}$ for $A \geq 2$ and $\tilde{g}$ are the components g_{AB} with $A, B \geq 2$. The inverse matrix, denoted by $g^{\mu\nu}$ and computed along the cone is

$$g^{\mu\nu} =|_{\mathcal{C}_p(\mathcal{U})} \begin{pmatrix} 0 & \nu^0 & 0 \\ \nu^0 & \left(\nu^0\right)^2 \left(-g_{00} + \|\nu^{-1}\|_{\tilde{g}}\right) & -\nu^0 \nu^{-1} \\ 0 & -\nu^0 \nu^{-1\,t} & \tilde{g}^{-1} \end{pmatrix},$$

where $\left(\tilde{g}^{-1}\right)^{AB} = g^{AB}$ for $A, B > 1$, $\nu^0 = g^{00}$ and ν^{-1} is the notation for the vector $(g^{12}, \ldots, g^{1\,n-1})$. The notation is partially motivated by the fact that along the cone $\tilde{g}^{-1}$ is the inverse matrix of $\tilde{g}$, $\nu^0 = \frac{1}{\nu_0}$ and $\nu^{-1} = \tilde{g}^{-1} \cdot \nu$. The added benefit of the coordinates that have just been constructed is the form of the conformal factor and its derivatives

$$\begin{aligned} \sigma =& -\left(x^0\right)^2 + 2x^0x^1 \\ d\sigma =& 2\left(x^1 - x^0\right) dx^0 + 2x^0 dx^1 \\ \operatorname{grad} \sigma =& 2\sum_i \left(\left(x^1 - x^0\right) g^{i0} + x^0 g^{i1}\right) \partial_{x^i} \\ \operatorname{Hess}^g \sigma =& 2\left(2dx^0 \odot dx^1 - dx^0 \odot dx^0\right) - 2\sum_{i,j} \left(\left(x^1 - x^0\right) \Gamma^0_{ij} + x^0 \Gamma^1_{ij}\right) dx^i \otimes dx^j. \end{aligned}$$

Also $\rho = -\frac{1}{n}\left(\operatorname{tr}^g \operatorname{Hess}^g \sigma + J\sigma\right)$ is written as

$$\rho = -\frac{2}{n}\left(2g^{01} - g^{00} - \left(x^1 - x^0\right)\Gamma^0 - x^0\Gamma^1\right) - \frac{-\left(x^0\right)^2 + 2x^0x^1}{2n(n-1)}\tau^g,$$

where $\Gamma^\kappa = \sum_{\mu,\nu} \Gamma^\kappa_{\mu\nu} g^{\mu\nu}$. All those quantities simplify if evaluated along the null cone $\{x^0 = 0\}$

$$\begin{aligned} d\sigma \quad &=|_{\mathcal{C}_p(\mathcal{U})} \; 2x^1 dx^0 \\ \operatorname{grad} \sigma \quad &=|_{\mathcal{C}_p(\mathcal{U})} \; 2\frac{x^1}{\nu_0}\partial_{x^1} \\ \operatorname{Hess}^g \sigma \quad &=|_{\mathcal{C}_p(\mathcal{U})} \; 2\left(2dx^0 \odot dx^1 - dx^0 \odot dx^0\right) - 2\sum_{i,j} x^1 \Gamma^0_{ij} dx^i \otimes dx^j \\ &= \quad 2\left(2dx^0 \odot dx^1 - dx^0 \odot dx^0\right) - \frac{x^1}{\nu_0}\sum_{i,j}\left(\partial_{x^i} g_{j1} + \partial_{x^j} g_{i1} - \partial_{x^1} g_{ij}\right) dx^i \otimes dx^j \\ &= \quad 2\left(2dx^0 \odot dx^1 - dx^0 \odot dx^0\right) + \frac{x^1}{\nu_0}\mathcal{L}_{\partial_{x^1}} g - 2\frac{x^1}{\nu_0}\sum_{i,j}\partial_{x^i} g_{j1} dx^i \odot dx^j \\ \rho \quad &=|_{\mathcal{C}_p(\mathcal{U})} \; -\frac{2}{n}\left(\frac{2}{\nu_0} - x^1\Gamma^0\right). \end{aligned}$$

For further calculations it is helpful to evaluate Γ^0 along the cone. Using Einstein notation, it reduces to the sum $g^{01}\left(\partial_i g_{j1} + \partial_j g_{i1} - \partial_1 g_{ij}\right) g^{ij}$. The components g_{1k} with $k \geq 1$ vanish along the cone. Hence tangent derivatives $\partial_l g_{1k}$ with $k, l \geq 1$ will also vanish. This gives

$$\begin{aligned} \Gamma^0 &=|_{\mathcal{C}_p(\mathcal{U})} \frac{1}{\nu_0^2}\partial_{x^0} g_{11} - \frac{1}{2\nu_0}\sum_{A,B\geq 2} g^{AB}\partial_{x^1} g_{AB} \\ &=: \frac{1}{\nu_0^2}\partial_{x^0} g_{11} - \frac{1}{2\nu_0}\operatorname{tr}^{\tilde{g}} \partial_1 \tilde{g}, \end{aligned}$$

where the term $\operatorname{tr}^{\tilde{g}} \partial_1 \tilde{g}$ is just a short notation for the sum. The Hessian along the cone may be written in matrix form and then is

$$\operatorname{Hess}^g \sigma =|_{\mathcal{C}_p(\mathcal{U})} \begin{pmatrix} -2 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{x^1}{\nu_0}\partial_1 \begin{pmatrix} g_{00} & \nu_0 & \nu \\ \nu_0 & 0 & 0 \\ \nu^t & 0 & \tilde{g} \end{pmatrix} + \frac{x^1}{\nu_0}\partial_0 \begin{pmatrix} g_{10} & \cdots & g_{1\,n-1} \\ \vdots & 0 & 0 \\ g_{1\,n-1} & 0 & 0 \end{pmatrix} + 2\frac{x^1}{\nu_0}\begin{pmatrix} \partial_0\nu_0 & \cdots & \partial_{n-1}\nu_0 \\ \vdots & 0 & 0 \\ \partial_{n-1}\nu_0 & 0 & 0 \end{pmatrix}.$$

As we started with an almost Einstein structure (M, g, σ), in the coordinates x the almost Einstein equation along the null cone ($\mathrm{Hess}^g\,\sigma + \rho g = 0$) reads

$$\begin{aligned}
0 \;&=|_{\mathcal{C}_p(U)} \; 2dx^0 \otimes dx^1 + 2dx^1 \otimes dx^0 - 2dx^0 \otimes dx^0 - \frac{x^1}{\nu_0}\sum_{i,j}\left(\partial_{x^i} g_{j1} + \partial_{x^j} g_{i1} - \partial_{x^1} g_{ij}\right) dx^i \otimes dx^j \\
&\quad -\frac{2}{n}\left(\frac{2}{\nu_0} - x^1\Gamma^0\right)\sum_{i,j} g_{ij} dx^i \otimes dx^j \\
&= \; 2dx^0 \otimes dx^1 + 2dx^1 \otimes dx^0 - 2dx^0 \otimes dx^0 - \frac{x^1}{\nu_0}\sum_{i,j}\left(\partial_{x^i} g_{j1} + \partial_{x^j} g_{i1} - \partial_{x^1} g_{ij}\right) dx^i \otimes dx^j \\
&\quad -\frac{2}{n\nu_0}\left(2 - \frac{x^1}{\nu_0}\partial_{x^0} g_{11} + \frac{x^1}{2}\operatorname{tr}^{\tilde g}\partial_1 \tilde g\right)\sum_{i,j} g_{ij} dx^i \otimes dx^j .
\end{aligned}$$

Prescribing the metric g on the cone clearly fixes the tangent derivatives $\mathcal{L}_{\partial_{x^i}} g$ along the cone for $i \geq 1$. In addition a couple of transversal derivatives $\partial_{x^0} g_{ij}$ are fixed by the almost Einstein equation and the components of $\tilde g$ have to fulfil some constraint equations. First consider the term $dx^0 \otimes dx^1$. The almost Einstein equation implies

$$\begin{aligned}
0 \;&=|_{\mathcal{C}_p(U)} \; 1 - \frac{x^1}{2\nu_0}\partial_{x^0} g_{11} - \frac{1}{n\nu_0}\left(2 - \frac{x^1}{\nu_0}\partial_{x^0} g_{11} + \frac{x^1}{2}\operatorname{tr}^{\tilde g}\partial_1 \tilde g\right)\nu_0 \\
&= \; \frac{n-2}{n} - \frac{n-2}{2n}\frac{x^1}{\nu_0}\partial_{x^0} g_{11} - \frac{x^1}{2n}\operatorname{tr}^{\tilde g}\partial_1 \tilde g \\
&= \; \frac{n-2}{2n}\left(2 - \frac{x^1}{\nu_0}\partial_{x^0} g_{11} + \frac{x^1}{2}\operatorname{tr}^{\tilde g}\partial_1 \tilde g\right) - \frac{x^1}{4}\operatorname{tr}^{\tilde g}\partial_1 \tilde g
\end{aligned}$$

On the one hand, provided $\tilde g$ is prescribed on the cone, this fixes the transversal derivative of g_{11} on the cone. On the other hand, substituting this equation to the almost Einstein equation in adapted coordinates gives

$$\begin{aligned}
0 =|_{\mathcal{C}_p(U)} \; & 2dx^0 \otimes dx^1 + 2dx^1 \otimes dx^0 - 2dx^0 \otimes dx^0 \\
& - \frac{x^1}{\nu_0}\sum_{i,j}\left(\partial_{x^i} g_{j1} + \partial_{x^j} g_{i1} - \partial_{x^1} g_{ij} + \frac{\operatorname{tr}^{\tilde g}\partial_1 \tilde g}{n-2} g_{ij}\right) dx^i \otimes dx^j
\end{aligned}$$

The term $dx^0 \otimes dx^i$ for $i \neq 1$ gives equations to the transversal derivative of ν_0 for $i = 0$

$$0 =|_{\mathcal{C}_p(U)} 2x^1\partial_0 \nu_0 + 2\nu_0 + x^1\left(-\partial_{x^1} g_{00} + \frac{\operatorname{tr}^{\tilde g}\partial_1 \tilde g}{n-2} g_{00}\right)$$

and to the transversal derivative of g_{1A} for $i = A \geq 2$

$$0 =|_{\mathcal{C}_p(U)} \partial_{x^0} g_{1A} + \partial_{x^A}\nu_0 - \partial_{x^1}\nu_A + \frac{\operatorname{tr}^{\tilde g}\partial_1 \tilde g}{n-2}\nu_A .$$

The coefficients of the terms $dx^1 \otimes dx^i$ for $i \geq 1$ vanish, since g_{1i} is constant along the cone and so all its tangent derivatives will vanish. The remaining equations then give an ordinary differential equation to $\tilde g$ that has to be fulfilled for almost Einstein metrics on the cone in the specific coordinates

$$0 =|_{\mathcal{C}_p(U)} \partial_{x^1}\tilde g - \frac{\operatorname{tr}^{\tilde g}\partial_{x^1}\tilde g}{n-2}\tilde g . \tag{5.28}$$

This equation is solved by any decomposition $\tilde g(x) =|_{\mathcal{C}_p(U)} f(x)\cdot G(x^2, \ldots, x^{n-1})$, where $f : \mathbb{R}^n \to \mathbb{R}$ is a non-vanishing smooth map and $G : \mathbb{R}^{n-2} \to \mathrm{Mat}_n(\mathbb{R})$ is a matrix-valued map with $G(\tilde x)$ positive definite and symmetric. In fact all solutions to the differential-algebraic Equation (5.28) are of this form (see appendix D).

Considering the asymptotic behaviour as $x^1 = r$ goes to zero gives further restrictions to the form of $\tilde g$. The coordinates ϑ appeared as a transformation of Morse coordinates φ in the preliminaries of Theorem 5.3.38. Conversely one may also write them as transformation of geodesic coordinates by observing

$$\vartheta = F \circ \zeta^{-1} \circ \tilde{\varphi} =: \tilde{F} \circ \tilde{\varphi},$$

which then defines the coordinate transformation $\tilde{F}$. Both transformations F and ζ are cone-preserving diffeomorphisms close to the identity. In particular we will need the properties $F(y) = y + \mathrm{o}(\|x\|)$ and $\zeta(x) = x + \mathrm{o}(\|x\|)$ such that we also have $\tilde{F}(x) = x + \mathrm{o}(\|x\|)$ or $d\tilde{F}(x) = \mathrm{id} + \mathcal{O}(\|x\|)$ for the Jacobian. We denote with g^{ϑ}_{ij} and $g^{\tilde{\varphi}}_{ij}$ the metrics in different coordinates. Then $g^{\vartheta} = \left(d\tilde{F}^{-1}\right)^t \cdot g^{\tilde{\varphi}} \cdot d\tilde{F}^{-1}$. A property of geodesic coordinates is the asymptotic behaviour $g^{\tilde{\varphi}}_{ij}(x) = \kappa\eta_{ij} + \mathcal{O}(\|x\|^2)$ with the Minkowski form η and a constant $\kappa \in \mathbb{R}$, as we have not started with normalised coordinates. By using the expansion for $d\tilde{F}$ this also provides $g^{\vartheta}_{ij}(x) = \kappa\eta_{ij} + \mathcal{O}(\|x\|^2)$. Following [CBCMG11b, section 4.5.] this gives rise to the asymptotic $\tilde{g}(x) =|_{\mathcal{C}_p(U)} \kappa \left(x^1\right)^2 \left(\Omega^{S^{n-2}}(x) + \mathcal{O}\left(\left(x^1\right)^2\right)\right)$ in coordinates x, where $\Omega^{S^{n-2}}$ is the round metric in the spherical coordinates x^A. The decomposition $\tilde{g}(x) =|_{\mathcal{C}_p(U)} f(x) \cdot G(x^2, \ldots, x^{n-1})$ then requires $G = \Omega^{S^{n-2}}$ and $f(x) = \kappa \left(x^1\right)^2 \left(1 + \mathcal{O}\left(\left(x^1\right)^2\right)\right)$.

Finally, we obtain a reduced form for the metric in coordinates x adapted to the null cone

$$g_{\mu\nu} =|_{\mathcal{C}_p(\mathcal{U})} \begin{pmatrix} g_{00} & \nu_0 & \nu \\ \nu_0 & 0 & 0 \\ \nu^t & 0 & f\Omega^{S^{n-2}} \end{pmatrix}, \tag{5.29}$$

where $g_{00} - \kappa$, $\nu_0 + \kappa$, ν and f are at least of order $\mathcal{O}\left(\left(x^1\right)^2\right)$ (see [CBCMG11b, section 4.5.] for the first components).

6 OUTLOOK

We will now summarise open problems that emerged while writing the thesis.

A first challenge is a more detailed attempt to generalise the conformal wave equations to higher even dimensions as it is only sketched in section 4.3. By introducing powers $\Delta^k \mathrm{P}$, $\Delta^k \mathrm{C}$, $\Delta^k \mathrm{W}$ and their derivatives as new variables to the system, we would like to have a more explicit expression of the obstruction tensor in terms of the new set of unknowns. The next step would be to provide initial data on the characteristic cone for that set of unknowns.

The adapted coordinates constructed in the latter section may provide a tool for the analysis of such an initial data problem. But at the moment it is only a tool that lacks an application, since the author is not an expert in partial differential equations. The matter of whether the advantage of eliminating the conformal factor that is gained by the disadvantage of loosing smoothness at the vertex really is of some use will have to be examined in future works.

On the way to the construction of adapted coordinates we got a lifting result for diffeomorphisms on the sphere and on the compact cylinder that are close to the identity map (Proposition 5.3.21 and 5.3.31). The problem may be generalised in the following way. Let M be a compact manifold with a Lie group G acting on it. Then consider a local C^m-diffeomorphism $f : M \to M$, which admits a map $g : M \to \underline{\mathfrak{g}}$ such that $f(x) = \exp(g(x)) \cdot x$. The map g is not even assumed to be continuous. The question is, what additional requirements on M, G and f suffice to ensure that g inherits the smoothness of f. A simple counterexample can be found for $M = S^1$, $G = SO(2)$, $\underline{\mathfrak{g}} = \mathbb{R}$. The local smooth diffeomorphism $f : e^{i\varphi} \mapsto e^{2\varphi}$ cannot be written as $f(x) = \exp(g(x)) \cdot x$ with a smooth map g, as $f(e^{i\varphi}) = e^{i\varphi} \cdot e^{i\varphi}$ would up to a constant $2\pi k$ imply $g\left(e^{i\varphi}\right) = \varphi$. Such a map cannot even be continuous.

Main considerations of the thesis start with almost Einstein structures instead of conformally compactified Einstein manifolds. Such structures correspond to parallel tractors in the tractor bundle over a conformal structure. The existence of an almost Einstein structure can then equivalently be treated as existence problem for parallel tractors. Parallel sections in the tractor bundle on the other hand can be approached via examination of the holonomy of the tractor connection. There actually is a lot of interest in conformal holonomy in Lorentzian signature. We have not followed this path at this time but it may lead to new examples of conformally compactified Einstein manifolds. Closely related to that issue is the property of the singularity set to decompose into a set of isolated point and null hypersurfaces. This corresponds to the curved orbit decomposition [ČGH14] of an almost Einstein structure. Now vice versa it would be interesting to identify the requirements of a manifold M^n without boundary that guarantee that it at least topologically admit such a disjoint decomposition, In particular a decomposition into open sets $\mathring{M}$, $(n-1)$-dimensional submanifolds Σ_c and isolated points Σ_d such that in addition $\partial\mathring{M} = \Sigma_c \cup \Sigma_d$, $\partial\Sigma_c = \Sigma_d$ and close to isolated points the union $\Sigma_c \cup \Sigma_d$ should have the topology of a double cone. Next question then would be what is needed to assure that this decomposition is compatible with a metric in the sense that it locally gives $\Sigma_c \cup \Sigma_d$ the causal structure of a null cone or null hypersurface.

A BASIC PROOFS

A.1 A SELECTION OF BASIC PROOFS

A set of selected claims will be proved now. The main reason that the proofs are separated from the remaining thesis is that the claims are well known or at least basic facts. Nevertheless the proofs do provide examples for the methods of calculation, which have been used throughout the thesis. We believe that at least some of them are not very common to all the readers.

Lemma 1.1.1. *Let (M, g) be a pseudo-Riemannian manifold, ∇ the Levi-Civita connection and $\omega \in \Omega^1(M)$ a 1-form on (M, g). Then the following Weitzenböck identity connects the Hodge Laplacian Δ_1 with the Bochner Laplacian Δ^∇.*

$$\Delta_1 \omega = \Delta^\nabla \omega + \mathrm{Ric}^\sharp(\omega)$$

Proof: Let $p \in M$ be an arbitrary point, $\{e_i\}$ a local frame and $\omega \in \Omega^1(M)$ then

$$\begin{aligned}
\delta d\omega(X) &\overset{(1.12)}{=} -2\sum_{i=1}^{n} \epsilon_i \left(\nabla_{e_i} P_A \nabla\omega\right)(e_i, X) \\
&= -\sum_{i=1}^{n} \epsilon_i \left(\nabla_{e_i}\nabla_{e_i}\omega\right)(X) + \sum_{i=1}^{n} \epsilon_i \left(\nabla_{e_i}\nabla_X\omega\right)(e_i) \\
&= \left(\Delta^\nabla \omega\right)(X) + \mathrm{tr}^g_{1,3}\left(\nabla\nabla\omega\right)(X).
\end{aligned}$$

The last calculation may be summarised as

$$\delta d\omega = \Delta^\nabla \omega + \mathrm{tr}^g_{1,3}\left(\nabla\nabla\omega\right). \tag{A.1}$$

On the other hand for $T \in \mathcal{T}^{p,0}M$ one has $\mathrm{tr}^g_{i,j} \nabla T = \nabla\, \mathrm{tr}_{i-1,j-1} T$ and thus

$$\begin{aligned}
d\delta\omega &\overset{(1.13)(1.12)}{=} \frac{1}{p} P_A\left(-\nabla\, \mathrm{tr}^g_{1,2} \nabla\omega\right) \\
&= -\frac{1}{p} P_A\left(\mathrm{tr}^g_{2,3} \nabla\nabla\omega\right)
\end{aligned} \tag{A.2}$$

The Weitzenböck identity then is the result of the following calculation

$$\begin{aligned}
d\delta\omega + \delta d\omega &\overset{(A.1),(A.2)}{=} \Delta^\nabla\omega + \mathrm{tr}^g_{1,3} \nabla\nabla\omega - \mathrm{tr}^g_{2,3} \nabla\nabla\omega \\
&= \Delta^\nabla\omega + \sum_{i=1}^{n} \epsilon_i \left(\left(\nabla\nabla\omega\right)(e_i, \cdot, e_i) - \left(\nabla\nabla\omega\right)(\cdot, e_i, e_i)\right) \\
&\overset{(1.17)}{=} \Delta^\nabla\omega + \sum_{i=1}^{n} \epsilon_i \left(R\left(e_i, \cdot\right)\omega\right)(e_i) \\
&= \Delta^\nabla\omega + \sum_{i=1}^{n} \epsilon_i g\left(e_i, R(e_i, \cdot)\omega^\sharp\right) \\
&\overset{(1.25)}{=} \Delta^\nabla\omega - \mathrm{Ric}(\omega^\sharp, \cdot).
\end{aligned}$$

■

Corollary 1.1.8. *Let T be a $(4,0)$-tensor with symmetries of the $(4,0)$-Riemann tensor and B be a symmetric $(2,0)$-tensor. Then any double metric trace of the tensor product of those two tensors will give* *an symmetric* $(2,0)$*-tensor, i.e.* $\operatorname{tr}^g_{i,j}(\operatorname{tr}_{k,l} T \otimes B)$ *is a symmetric tensor for all* $k \neq l \in \{1,\dots,6\}$ *and* $i \neq j \in \{1,\dots,4\}$.

Proof: This fact is clear if the traces are taken with respect to antisymmetric arguments of T as for example $\operatorname{tr}^g_{1,2}\operatorname{tr}^g_{2,3} T \otimes S$. The result would vanish everywhere. In place of the remaining contractions consider $\operatorname{tr}_{1,4}\operatorname{tr}_{1,5} T \otimes S$. Using the first Bianchi identity one obtains

$$\begin{aligned}
\operatorname{tr}^g_{1,4}\operatorname{tr}^g_{1,5} T \otimes S(X,Y) &= \sum_{i,j} \epsilon_i\epsilon_j T(e_i,X,Y,e_j)S(e_i,e_j) \\
&= -\sum_{i,j} \epsilon_i\epsilon_j T(Y,e_i,X,e_j)S(e_i,e_j) \\
&= \sum_{i,j} \epsilon_i\epsilon_j T(e_i,Y,X,e_j)S(e_i,e_j) \\
&= \operatorname{tr}^g_{1,4}\operatorname{tr}^g_{1,5} T \otimes S(Y,X)
\end{aligned}$$

■

Lemma 1.1.9. *Let* M *be a symmetric* $(2,0)$*-tensor, then the Kulkarni-Nomizu product* $M \owedge g$ *fulfils*

$$\begin{aligned}
\operatorname{tr}_{1,3}(M \owedge g) &= \operatorname{tr}(M)g + (n-2)M \\
(\operatorname{div} M \owedge g)(X,Y,Z) &= (\operatorname{div} M)(Y)g(X,Z) - (\operatorname{div} M)(Z)g(X,Y) \\
&\quad + (\nabla_Z M)(X,Y) - (\nabla_Y M)(X,Z).
\end{aligned}$$

Proof: Using an orthonormal frame $\{e_i\}$ the first equations is a consequence of

$$\begin{aligned}
(\operatorname{tr}_{1,3} M \owedge g)(X,Y) &= (\operatorname{tr} M)g(X,Y) + nM(X,Y) \\
&\quad - \sum_{i\in\{1,\dots,n\}} \epsilon_i \left(M(Y,e_i)g(X,e_i) + M(X,e_i)g(Y,e_i)\right) \\
&= (\operatorname{tr} M)g(X,Y) + nM(X,Y) - \left(M(Y,X) + M(X,Y)\right).
\end{aligned}$$

The second equation is gained using compatibility of the Levi-Civita connection with the metric. So only derivatives of M remain if $\nabla M \owedge g$ is calculated

$$\begin{aligned}
(\nabla M \owedge g)(U,V,X,Y,Z) &= (\nabla M)(U,V,Y)g(X,Z) + (\nabla M)(U,X,Z)g(V,Y) \\
&\quad - (\nabla M)(U,V,Z)g(X,Y) - (\nabla M)(U,X,Y)g(V,Z) \\
&= (\nabla_U M)(V,Y)g(X,Z) + (\nabla_U M)(X,Z)g(V,Y) \\
&\quad - (\nabla_U M)(V,Z)g(X,Y) - (\nabla_U M)(X,Y)g(V,Z).
\end{aligned}$$

Hence one obtains

$$\begin{aligned}
(\operatorname{div} M \owedge g)(X,Y,Z) &= -\sum_i \left(\epsilon_i (\nabla M \owedge g)(e_i,e_i,X,Y,Z)\right) \\
&= -\sum_i \left(\epsilon_i(\nabla_{e_i} M)(e_i,Y)g(X,Z) - \epsilon_i(\nabla_{e_i} M)(e_i,Z)g(X,Y)\right) \\
&\quad - \sum_i \Big((\nabla_{\epsilon_i g(e_i,Y)e_i} M)(X,Z) - (\nabla_{\epsilon_i g(e_i,Z)e_i} M)(X,Y)\Big) \\
&= (\operatorname{div} M)(Y)g(X,Z) - (\operatorname{div} M)(Z)g(X,Y) \\
&\quad + (\nabla_Z M)(X,Y) - (\nabla_Y M)(X,Z)
\end{aligned}$$

■

Lemma 1.1.11. *The Cotton and Bach tensors have the following properties*

$$\begin{aligned}
(\operatorname{div}_2 \mathrm{C})(X,Y) &= (\operatorname{div}_2 \mathrm{C})(Y,X) \\
0 &= \operatorname{div} \mathrm{C} \\
0 &= \mathrm{C}(X,Y,Z) + \mathrm{C}(Y,Z,X) + \mathrm{C}(Z,X,Y) \\
0 &= \mathfrak{B}(X,Y) - \mathfrak{B}(Y,X).
\end{aligned}$$

Namely the divergence of the Cotton tensor is symmetric and vanishes if taken in the first argument, the Cotton tensor fulfils the first Bianchi identity and the Bach tensor is symmetric. In addition the Cotton tensor ist totally trace-free due to the same property of the Weyl tensor.

Proof: The Bianchi identity for the Cotton tensor is a direct consequence of the Bianchi identity satisfied by the Weyl tensor. Symmetry of $\mathrm{div}_2\,\mathrm{C}$ is a consequence of

$$\begin{aligned}
-(n-3)\left(\mathrm{div}_2\,\mathrm{C}(X,Y)-\mathrm{div}_2\,\mathrm{C}(Y,X)\right) &= (n-3)\sum_i \epsilon_i\left((\nabla\,\mathrm{C})(e_i,X,e_i,Y)-(\nabla\,\mathrm{C})(e_i,Y,e_i,X)\right)\\
&= -\sum_{i,j}\epsilon_i\epsilon_j\left((\nabla\nabla\,\mathrm{W})(e_i,e_j,e_j,X,e_i,Y)-(\nabla\nabla\,\mathrm{W})(e_i,e_j,e_j,Y,e_i,X)\right)\\
&= -\sum_{i,j}\epsilon_i\epsilon_j\left(\mathrm{R}(e_i,e_j)\,\mathrm{W}\right)(e_j,X,e_i,Y)\\
&\overset{(1.18)}{=} \sum_{i,j}\epsilon_i\epsilon_j\,\mathrm{W}(\mathrm{R}(e_i,e_j)e_j,X,e_i,Y)+\sum_{i,j}\epsilon_i\epsilon_j\,\mathrm{W}(e_j,\mathrm{R}(e_i,e_j)X,e_i,Y)\\
&\quad+\sum_{i,j}\epsilon_i\epsilon_j\,\mathrm{W}(e_j,X,\mathrm{R}(e_i,e_j)e_i,Y)+\sum_{i,j}\epsilon_i\epsilon_j\,\mathrm{W}(e_j,X,e_i,\mathrm{R}(e_i,e_j)Y)\\
&= \sum_{i,j,k}\epsilon_i\epsilon_j\epsilon_k\,\mathrm{R}(e_k,e_j,e_i,e_j)\,\mathrm{W}(e_k,X,e_i,Y)+\sum_{i,j,k}\epsilon_i\epsilon_j\epsilon_k\,\mathrm{R}(e_k,X,e_i,e_j)\,\mathrm{W}(e_j,e_k,e_i,Y)\\
&\quad+\sum_{i,j,k}\epsilon_i\epsilon_j\epsilon_k\,\mathrm{R}(e_k,e_i,e_i,e_j)\,\mathrm{W}(e_j,X,e_k,Y)+\sum_{i,j,k}\epsilon_i\epsilon_j\epsilon_k\,\mathrm{R}(e_k,Y,e_i,e_j)\,\mathrm{W}(e_j,X,e_i,e_k)\\
&\overset{(1.38)}{=} \sum_{i,j,k}\epsilon_i\epsilon_j\epsilon_k\,\mathrm{W}(e_k,X,e_i,e_j)\,\mathrm{W}(e_j,e_k,e_i,Y)+\sum_{i,j,k}\epsilon_i\epsilon_j\epsilon_k\,\mathrm{W}(e_k,Y,e_i,e_j)\,\mathrm{W}(e_j,X,e_i,e_k)\\
&\quad+\sum_{i,j,k}\epsilon_i\epsilon_j\epsilon_k(\mathrm{P}\owedge g)(e_k,X,e_i,e_j)\,\mathrm{W}(e_j,e_k,e_i,Y)\\
&\quad+\sum_{i,j,k}\epsilon_i\epsilon_j\epsilon_k(\mathrm{P}\owedge g)(e_k,Y,e_i,e_j)\,\mathrm{W}(e_j,X,e_i,e_k)\\
&= +\sum_{i,k}\epsilon_i\epsilon_k\mathrm{P}(e_i,e_k)\,\mathrm{W}(X,e_k,e_i,Y)-\sum_{j,k}\epsilon_j\epsilon_k\mathrm{P}(e_j,e_k)\,\mathrm{W}(e_j,X,Y,e_k)\\
&= 0,
\end{aligned}$$

where $\{e_i\}$ is a local orthonormal frame. A similar argument holds for the second equation. Using Equation (1.22) for the dot-product one obtains

$$\begin{aligned}
\left(\mathrm{tr}_{1,2}\,\mathrm{tr}_{1,3}(\mathrm{R}_{vf}\cdot\mathrm{W})\right)(X,Y) &\overset{(1.17)}{=} \sum_{i,j=1}^{n}\epsilon_i\epsilon_j\left((\nabla\nabla\mathrm{W})(e_i,e_j,e_i,e_j,X,Y)-(\nabla\nabla\mathrm{W})(e_j,e_i,e_i,e_j,X,Y)\right)\\
&= -2\left(\mathrm{tr}_{1,2}\,\mathrm{tr}_{2,3}(\nabla\nabla\mathrm{W})\right)(X,Y)\\
&= -2\left(\mathrm{div}\,\mathrm{div}\,\mathrm{W}\right)(X,Y)
\end{aligned}$$

On the other hand using Equation (1.18) the left-hand side also equals

$$\begin{aligned}
\left(\mathrm{tr}_{1,2}\,\mathrm{tr}_{1,3}(\mathrm{R}_{vf}\cdot\mathrm{W})\right)(X,Y) &= \sum_{i,j}\epsilon_i\epsilon_j\left(\mathrm{R}(e_i,e_j)\,\mathrm{W}\right)(e_i,e_j,X,Y)\\
&= -\sum_{i,j}\epsilon_i\epsilon_j\left(\mathrm{W}(\mathrm{R}(e_i,e_j)e_i,e_j,X,Y)+\mathrm{W}(e_i,\mathrm{R}(e_i,e_j)e_j,X,Y)\right)\\
&\quad-\sum_{i,j}\epsilon_i\epsilon_j\left(\mathrm{W}(e_i,e_j,\mathrm{R}(e_i,e_j)X,Y)+\mathrm{W}(e_i,e_j,X,\mathrm{R}(e_i,e_j)Y)\right)\\
&= -\sum_{i,j,k}\epsilon_i\epsilon_j\epsilon_k\,\mathrm{W}(e_i,e_j,e_k,Y)(\mathrm{P}\owedge g)(e_i,e_j,e_k,X)\\
&\quad+\sum_{i,j,k}\epsilon_i\epsilon_j\epsilon_k\,\mathrm{W}(e_i,e_j,e_k,X)(\mathrm{P}\owedge g)(e_i,e_j,e_k,Y).\\
&= -\sum_{i,k}\epsilon_i\epsilon_k\,\mathrm{W}(e_i,X,e_k,Y)\mathrm{P}(e_i,e_k)+\sum_{j,k}\epsilon_j\epsilon_k\,\mathrm{W}(X,e_j,e_k,Y)\mathrm{P}(e_j,e_k)\\
&\quad+\sum_{i,k}\epsilon_i\epsilon_k\,\mathrm{W}(e_i,Y,e_k,X)\mathrm{P}(e_i,e_k)-\sum_{j,k}\epsilon_j\epsilon_k\,\mathrm{W}(Y,e_j,e_k,X)\mathrm{P}(e_j,e_k)\\
&= 0,
\end{aligned}$$

which proves the second property. By using Equation (1.44) for the Bach tensor one has

$$\begin{aligned}&\mathfrak{B}(X,Y)-\mathfrak{B}(Y,X)=\\&\qquad \operatorname{tr}_{1,3}(\operatorname{tr}_{1,3}\mathrm{P}\otimes \mathrm{W})(X,Y)-\operatorname{tr}_{1,3}(\operatorname{tr}_{1,3}\mathrm{P}\otimes \mathrm{W})(Y,X)+(\operatorname{div}_2\mathrm{C})(X,Y)-(\operatorname{div}_2\mathrm{C})(Y,X).\end{aligned}$$

The first part vanishes due to Corollary 1.1.8, the second due to Equation (1.47). Hence the Bach tensor is symmetric. ■

Lemma 1.1.16. *Let $X:\mathbb{R}^n\supset U\to\mathbb{R}^n$ be a smooth vector field, such that for at least one component $\left|X^k(x)\right|>\delta>0$ for all $x\in U$. Let $p\in U$ be a point and consider $\epsilon>0$ such that the open ball $B_{2\epsilon}(p)$ is a subset of U. Let be $B:=B_\epsilon(p)\subset U$, then every maximal integral curve starting within B ($\gamma(t)\in B$ for some $t\in\mathbb{R}$) will leave B within finite time in both directions.*

Proof: [Waloo, Theorem 6.VII] will be applied for the proof. Define $D:=\mathbb{R}\times U$. Then the map $f:D\ni(x,y)\mapsto X(y)\in\mathbb{R}^n$ is locally Lipschitz continuous. The initial value problem $y'=f(x,y)$ with $y(\xi)=\eta\in U$ has a unique maximal solution φ such that $(x,\varphi(x))$ is arbitrarily close to the boundary of D in both directions. Formally, treating infinity as points $-\infty$ and ∞, the boundary is written as $\partial D=\{-\infty,+\infty\}\times\bar{U}\cup\mathbb{R}\times\partial U=:\partial D_1\cup\partial D_2$. If the solution is arbitrarily close to ∂D_2 it will clearly leave any compact set within U. Arbitrarily close to the formally defined set $\partial_1=\{-\infty,+\infty\}\times\bar{U}$ means that φ is defined for all $x\in(\xi,\infty)$ or for all $x\in(-\infty,\xi)$. In particular x is unbounded in at least one direction. As φ solves the equation $\varphi'=X(\varphi)$, it remains to consider the component φ^k, for which $\left|\varphi'^k\right|>\delta$ is bounded from below. Hence the solution fulfils

$$\begin{aligned}\varphi^k(x)-\eta^k&=\int_\xi^x X^k(\varphi(s))ds\\ \implies\left|\varphi^k(x)-\eta^k\right|&>|x-\xi|\delta\end{aligned}$$

for all x in the domain of φ. Since φ is defined for all x in at least one direction, the component φ^k in that direction is unbounded and hence φ will leave any bounded region in U. In particular it will leave $B_\epsilon(p)$. ■

Lemma 1.2.3. *Let $\gamma:I\to M$ be a pregeodesic satisfying $\left(\nabla_{\dot\gamma}\dot\gamma\right)(t)=c(t)\dot\gamma(t)$ for some smooth function c. Then there exists a reparametrisation $h:I'\to I$, such that $\tilde\gamma:=\gamma\circ h$ is a geodesic in M.*

Proof: The proof follows the outline given in [O'N83]. Consider a reparametrisation $h:I'=(s_0,s_1)\to I=(t_0,t_1)$ and let $\tilde\gamma:=\gamma\circ h$. In the following dotted quantities are in some sense derivatives with respect to I, while primed ones are taken with respect to I'. Consequently tangent vectors to $\tilde\gamma$ will be denoted $\tilde\gamma'$.
One first shows the equivalence

$$\nabla_{\tilde\gamma'}\tilde\gamma'=0\quad\Longleftrightarrow\quad h''+h'^2\cdot(c\circ h)=0. \tag{A.3}$$

For the moment assume γ not to have self-intersections and let $H:\operatorname{im}(\gamma)\to\mathbb{R}$ be the map defined by $H(\gamma(h(s))):=h'(s)$. It is well defined due to non-vanishing of $d\gamma$. h is a reparametrisation and so $\left(\nabla_{\dot\gamma}H\right)(h(s))=\frac{d}{dt}\Big|_{t=h(s)}H(\gamma(t))=\frac{d}{dt}\Big|_{t=h(s)}h'\left(h^{-1}(t)\right)=\frac{h''(s)}{h'(s)}$. This is used in the following calculation.

$$\begin{aligned}\left(\nabla_{\tilde\gamma'}\tilde\gamma'\right)(s)&=\left(\nabla_{(\gamma\circ h)'}(\gamma\circ h)'\right)(s)\\&=h'(s)\cdot\nabla_{\dot\gamma(h(s))}\left(h'(s)\cdot\dot\gamma(h(s))\right)\\&=h'(s)\left[\dot\gamma(h(s))\cdot\left(\nabla_{\dot\gamma}H\right)(h(s))+h'(s)\cdot\left(\nabla_{\dot\gamma}\dot\gamma\right)(h(s))\right]\\&=h'(s)\dot\gamma(h(s))\left[\frac{h''(s)}{h'(s)}+h'(s)\cdot c(h(s))\right]\\&=\dot\gamma(h(s))\left[h''(s)+h'^2(s)\cdot c(h(s))\right]\end{aligned}$$

The equivalence follows from the requirement $\dot{\gamma} \neq 0$. If γ is self-intersecting, the interval can be split into overlapping parts, where there are no self-intersections within the segments. Since the equivalence holds for every segment, it holds all over I.

Solving the ordinary differential equation (ODE) $h'' + c \circ h \cdot h'^2 = 0$ in (A.3) is the second step. As long as h' does not vanish one has

$$h'' + c \circ h \cdot h'^2 = 0 \quad \overset{h' \neq 0}{\Longleftrightarrow} \quad \frac{d}{ds}\left(\log|h'| + C \circ h\right) = 0$$

where $C(t) := \int_{t_0}^{t} c(x)dx$ is an Integral of c. Absorbing any constant term into the integral C, it suffices to solve the ODE

$$h' = \exp\left(-C \circ h\right)$$

with initial condition $h(s_0) = t_0$. $h'(s_0)$ is fixed by adding a constant term to C. Here only positive values of h' are considered. The ODE is solved by separation of variables leading to an equation, which defines h by

$$\int_{t_0}^{h(s)} \exp(C(x))dx = s - s_0. \tag{A.4}$$

The integral is strictly monotonic, continuous on the interval (t_0, t_1) and therefore has an inverse map. Consequently Equation (A.4) admits a solution $h : (s_0, s_1) \to (t_0, t_1)$ which then solves (A.3). ∎

Lemma A.1.1 (1.2.4.). *Let $\mathcal{U} \subset M$ be a convex open set and $\gamma : [0, t_0) \to \mathcal{U}$ a geodesic, such that the limit $\lim_{t \to t_0} \gamma(t) \in \mathcal{U}$ exists. Then $t_0 < \infty$ and γ is extendible to the closed interval $[0, t_0]$.*

Proof: Denote $q := \gamma(0)$, $p := \lim_{t \to t_0} \gamma(t) \in \mathcal{U}$ and $\mathfrak{U}_q := \exp_q^{-1}(\mathcal{U}) \subset T_pM$. Then $\exp_q : \mathfrak{U}_q \to \mathcal{U}$ is a diffeomorphism. Since γ is a radial geodesic in q, there is a $X_0 \in \mathfrak{U}_q$ such that $\gamma(t) = \exp_q(X_0 t)$. Let be $X(t) := X_0 t$. Since $\exp_q^{-1}$ is continuous on $\mathfrak{U}$ it commutes with taking the limit $t \to t_0$ and one gets

$$\begin{aligned} \lim_{t \to t_0} X(t) &= \exp_q^{-1}\left(\lim_{t \to t_0} \gamma(t)\right) \\ &= \exp_q^{-1}(p). \end{aligned}$$

The limit exists in $\mathfrak{U}$, since p is an element in the convex set $\mathcal{U}$ and therefore t must not be unbounded and hence γ can be extended to a geodesic on the interval $[0, t_0]$ with $\gamma(t_0) = p$. ∎

Lemma A.1.2 (1.2.5.). *Let $p \in M$ be a point and $X \in T_pM$. Consider the geodesic defined by $\gamma(t) = \exp_p(tX)$. If γ is a closed geodesic with $\gamma(t_0) = \gamma(t_0 + \alpha)$ and $\dot{\gamma}(t_0 + \alpha) = \beta\dot{\gamma}(t_0))$ for $\alpha, \beta \in \mathbb{R}^+ \setminus \{0\}$ then*

(i) $t_0 < \alpha\frac{\beta}{1-\beta}$ for $\beta \in (0, 1)$ and there is no restriction to t_0 if $\beta \geq 1$.

(ii) $\exists t \in [t_0, t_0 + \alpha]$ with $\gamma(t) = p$.

Proof: The first part of the statement is a direct conclusion of Corollary 1.2.2, since the interval therein must at least be defined for $t = 0$ with $\gamma(0) = p$.

For the second part consider the geodesic defined by

$$\eta(t) := \gamma\left(\beta t + (1 - \beta)t_0 - \alpha\beta\right).$$

The initial data at $t_0 + \alpha$ are $\dot{\eta}(t_0 + \alpha) = \beta\dot{\gamma}(t_0) = \dot{\gamma}(t_0 + \alpha)$ and $\eta(t_0 + \alpha) = \gamma(t_0 + \alpha)$. Uniqueness of geodesics then implies

$$\gamma(t) = \gamma\left(\beta t + (1 - \beta)t_0 - \alpha\beta\right)$$

for all t in the domain of γ. Now consider I to be inextendible, then by Corollary 1.2.2 the interval limits depends on the value of β. For $\beta = 1$ the geodesic clearly is periodic with period

α. Since p is in the period containing 0 it is contained in every period and in particular it is contained in the image of $[t_0, t_0+\alpha]$. For $\beta \neq 1$ consider the map

$$\begin{array}{rccc} f: & \mathbb{R} & \rightarrow & I \\ & s & \mapsto & -\log(\beta)\beta^s + t_0 - \frac{\beta}{1-\beta}\alpha \end{array}$$

where I is the inextendible interval defined in Corollary 1.2.2. f is a smooth diffeomorphism onto the domain of γ. In addition

$$f(s+1) = \beta f(s) + (1-\beta)t_0 - \alpha\beta. \tag{A.5}$$

Therefore $\gamma(f(s)) = \gamma(f(s+1))$ for all $s \in \mathbb{R}$. Consequently $\gamma \circ f$ is periodic with period 1. Since there is an $r_0 \in \mathbb{R}$ such that $f(r_0) = 0$ and hence $\gamma \circ f(r_0) = p$, p is reached in every period. Moreover, there is an $r_1 \in \mathbb{R}$, such that $f(r_1) = t_0 + \alpha$. From (A.5) one gets $f(r_0+1) = t_0$ and therefore there is an $\tilde{r} \in [r_1, r_1+1]$ and hence $f(\tilde{r}) \in [t_0, t_0+\alpha]$, such that $\gamma(f(\tilde{r})) = p$. ∎

Proposition 1.2.8. *Let $\mathcal{U}$ be a normal neighbourhood of $p \in M$ and $\mathfrak{U} := \exp_p^{-1}(\mathcal{U})$. Then one has*

$$I^+(p,\mathcal{U}) = \exp_p\left(\mathfrak{T}_p^{\uparrow} \cap \mathfrak{U}\right).$$

Proof: Let $\mathfrak{H} := \mathfrak{T}_p^{\uparrow} \cap \mathfrak{U}$. It suffices to show $\exp_p(\mathfrak{H}) \subset I^+(p,\mathcal{U})$ and $I^+(p,\mathcal{U}) \subset \exp_p(\mathfrak{H})$.

For the first relation consider $X \in \mathfrak{H}$. Then $\gamma(t) := \exp_p(tX)$ is a future-directed timelike curve from p to $\exp_p(X)$ in $\mathcal{U}$ such that $p \lll_{\mathcal{U}} \exp_p(X)$ and hence $\exp_p(X) \in I^+(p,\mathcal{U})$.

For the second relation let $q \in I^+(p,\mathcal{U})$ be a point and assume $q \notin \exp_p(\mathfrak{H})$. Using the definition of $I^+(p,\mathcal{U})$, there is a piecewise smooth future-directed curve $\gamma : I \to \mathcal{U}$ with $\gamma(0) = p$ and $\gamma(1) = q$. Let $X(t) := \exp_p^{-1}(\gamma(t))$ be the corresponding curve in $\mathfrak{U}$. Then $\dot{X}(t) = \left(d\exp_p^{-1}\right)_{\gamma(t)}(\dot{\gamma}(t))$ is a future-directed timelike vector at $t = 0$. As long as $X(t) \in \mathfrak{T}_p^{\uparrow} \cap \mathfrak{U}$, $\left(d\exp_p\right)_{X(t)}(X(t))$ defines the time orientation of $T_{\gamma(t)}M$ at $\gamma(t) = \exp_p(X(t))$. The Gauß lemma then gives $g_p\left(\dot{X}(t), X(t)\right) = g_{\gamma(t)}\left(\dot{\gamma}(t), \left(d\exp_p\right)_{X(t)}(X(t))\right) < 0$. So in particular $g_p\left(X(t), X(t)\right)$ is strictly decreasing. Differentiating $g_p(X(t), X(t))$ twice yields that $g_p(X(0), X(0)) = 0$ is a local minimum and hence $g_p(X(t), X(t)) < 0$ at least for small t. On the other hand $X(t)$ must not intersect the cone $\mathfrak{C}_p$. Otherwise the Gauß lemma would provide $g_p(X(t), X(t)) = 0$ for that point, which is in contrast to the strict decreasing of $\|X\|^2_{g_p}$. ∎

Proposition 1.2.6. *Consider (M, g) to be a time-oriented Lorentzian manifold, $p \in M$, $\mathcal{U}$ a normal neighbourhood of p and $\mathcal{C}_p(\mathcal{U})$ the geodesic null cone in p. Furthermore let $\mathcal{N} \in \mathfrak{X}(\mathcal{U})$ be a vector field on U with the following properties:*

$$\begin{aligned} \mathcal{N}|_{\mathcal{U}\setminus\{p\}} &\neq 0 \\ \|\mathcal{N}\|^2\Big|_{\mathcal{C}_p(\mathcal{U})} &= 0. \\ \mathcal{N}_x \in T_x\mathcal{C}_p(\mathcal{U}) & \quad \textit{for} \quad x \in \mathcal{C}_p(\mathcal{U}) \end{aligned}$$

In particular $\mathcal{N}$ defines the isotropic direction on the tangent space of $\mathcal{C}_p(\mathcal{U}) \setminus \{p\}$. Let $T \in \mathcal{T}^{(p,0)}M$ be a tensor that is annihilated by $\mathcal{N}$ along the null cone $\mathcal{C}_p(U) \setminus \{p\}$, i.e.

$$T(\mathcal{N}, \cdot, \ldots, \cdot) =|_{\mathcal{C}_p(U)} 0.$$

Then T vanishes at p

$$T_p = 0.$$

Proof: It suffices to evaluate T_p on a basis of T_pM. Therefore choose the null basis $\{n_i\}$ in p as constructed in remark 1.2.1 and extend it to a geodesic null frame $\{N_i\}$ on the normal neighbourhood $\mathcal{U}$ by parallel transport along geodesics. Furthermore define the geodesics

$$\gamma_i : [0,\epsilon) \to \mathcal{C}_p \qquad \begin{aligned} \gamma_i(0) &= p \\ \dot\gamma_i(0) &= n_i \end{aligned}$$

for $i = 1, \ldots, n$. The image of each null geodesic γ_i is in $\mathcal{C}_p(\mathcal{U})$, as the geodesic null cone locally is generated by all null geodesics originating in p. Besides the tangent vector $\dot\gamma_i(t)$ coincides with the base vector N_i at $\gamma_i(t)$ by construction, i.e. $\dot\gamma_i(t) = (N_i)_{\gamma_i(t)}$. Let now $\mathcal{N}_i \in \Gamma(\gamma_i^* TM)$ be the restriction of $\mathcal{N}$ to a vector field along γ_i and denote $\mathcal{N}_i(t) := \mathcal{N}_{\gamma_i(t)}$.

Non-vanishing of $\mathcal{N} \neq 0$ away from p implies that the isotropic direction of $T_{\gamma_i(t)}\mathcal{C}_p(\mathcal{U})$ is determined by $\mathcal{N}_i(t)$. On the other hand it is also determined by $N_i(t) := (N_i)_{\gamma_i(t)}$. This implicitly defines maps $f_i : (0,\epsilon) \to \mathbb{R} \setminus \{0\}$ through

$$N_i(t) =: f_i(t) \cdot \mathcal{N}_i(t).$$

Applying all these considerations to the evaluation of T away from the origin, one obtains for $t \neq 0$

$$\begin{aligned} T_{\gamma_i(t)}(N_i(t), \cdot, \ldots, \cdot) &= f_i(t) T_{\gamma_i(t)}(\mathcal{N}_i(t), \cdot, \ldots, \cdot) \\ &= 0. \end{aligned}$$

By continuity of the arguments on the left-hand side, this must hold for $t = 0$ as well, i.e.

$$\begin{aligned} 0 &= T_{\gamma_i(0)}(N_i(0), \cdot, \ldots, \cdot) \\ &= T_p(n_i, \cdot, \ldots, \cdot). \end{aligned}$$

By construction $\{n_i\}$ is a null basis in p and hence the claim follows. ■

Lemma 1.4.3. *Let (M, g) be a pseudo-Riemannian manifold of dimension n. Consider a conformal change $\tilde g = \sigma^{-2} g$ of the metric. The following transformation laws holds for the Levi-Civita connection*

$$\tilde\nabla_X Y = \nabla_X Y - \sigma^{-1}\left(X(\sigma)Y + Y(\sigma)X - g(X,Y)\,\mathrm{grad}\,\sigma\right) \tag{A.6}$$

$$\tilde\nabla_X \omega = \nabla_X \omega - \sigma^{-1}\Big(\omega(\mathrm{grad}\,\sigma)X^\flat - X(\sigma)\omega - \omega(X)d\sigma\Big). \tag{A.7}$$

where all objects on the right-hand side are taken with respect to the unchanged metric g.

Proof: The conformal transformation rule for the Levi-Civita connection of $\tilde g$ can for example be calculated by using the Koszul formula and by using the fact that $X(\tilde g(Y,Z)) = \sigma^{-2}X(g(Y,Z)) - 2\sigma^{-1}X(\sigma)\tilde g(Y,Z)$. Then

$$\begin{aligned} \tilde g(\tilde\nabla_X Y, Z) &= \frac{1}{2}\Big(X(\tilde g(Y,Z) + Y(\tilde g(Z,X) - Z(\tilde g(X,Y) \\ &\quad + \tilde g(Z,[X,Y]) + \tilde g(Y,[Z,X]) - \tilde g(X,[Y,Z])\Big) \\ &= \frac{1}{2}\Bigg(\sigma^{-2}\Big(X(g(Y,Z) + Y(g(Z,X) - Z(g(X,Y) \\ &\quad + g(Z,[X,Y]) + g(Y,[Z,X]) - g(X,[Y,Z])\Big) \\ &\quad - 2\sigma^{-1}\Big(X(\sigma)\tilde g(Y,Z) + Y(\sigma)\tilde g(Z,X) - Z(\sigma)\tilde g(X,Y)\Big)\Bigg) \\ &= \sigma^{-2}g(\nabla_X Y, Z) - \sigma^{-1}\big(\tilde g(X(\sigma)Y, Z) + \tilde g(Y(\sigma)X, Z) - \sigma^{-2}g(\mathrm{grad}\,\sigma, Z)g(X,Y)\big) \\ &= \tilde g\Big(\nabla_X Y - \sigma^{-1}\big(X(\sigma)Y + Y(\sigma)X - g(X,Y)\,\mathrm{grad}\,\sigma\big), Z\Big). \end{aligned}$$

The corresponding transformation rule for the Levi-Civita connection applied to forms then can be calculated by the use of dualisation. Dualisation commutes with the Levi-Civita connection and in particular $\tilde{\nabla}_X\omega = \left(\tilde{\nabla}_X\omega^{\tilde{\sharp}}\right)^{\tilde{\flat}}$. The transformation rule for vector fields (1.101) can be applied to $\omega^{\tilde{\sharp}}$. Using $X^{\tilde{\flat}} = \sigma^{-2}X^{\flat}$ and $\eta^{\tilde{\sharp}} = \sigma^2\eta^{\sharp}$ then gives the rule for forms. ∎

Lemma A.1.3. *Let $f \in C^\infty(M)$ be a smooth function on (M,g) with $g(\operatorname{grad} f, \operatorname{grad} f) = const.$, then $\nabla_{\operatorname{grad} f} \operatorname{grad} f = 0$.*

Proof: Let $X \in \mathfrak{X}(M)$ be an arbitrary vector field, then having in mind Equation (1.3) for the Hessian of σ one gets

$$\begin{aligned} g\left(\nabla_{\operatorname{grad} f} \operatorname{grad} f, X\right) &= g\left(\nabla_X \operatorname{grad} f, \operatorname{grad} f\right) \\ &= \frac{1}{2} X\left(g(\operatorname{grad} f, \operatorname{grad} f)\right) \\ &= 0 \end{aligned}$$

and hence the claim follows. ∎

Lemma A.1.4. *Let $g = d\sigma^2 + g_\sigma$ be a metric on $\Sigma \times I$, where σ parametrises the interval I and g_σ is a curve of metrics on Σ. Let ∇ be the Levi-Civita connection of g and X, Y, Z be vector fields that are tangent to the slices $t \times \Sigma$ everywhere. Then*

$$\nabla_{\operatorname{grad}\sigma} \operatorname{grad}\sigma = 0 \tag{A.8}$$
$$\left(\nabla_Z g_\sigma\right)(X,Y) = 0 \tag{A.9}$$

Proof: Vanishing of $\nabla_{\operatorname{grad}\sigma} \operatorname{grad}\sigma = 0$ is a consequence of the assumed foliation. To show this claim one first observes that by definition $\operatorname{grad}\sigma$ is the normal vector field to the slices $\{\sigma = const.\}$. Now consider Y to be a local vector field that is tangent to the slices. Then $g(\operatorname{grad}\sigma, Y) \equiv 0$ and hence $g\big(\nabla_{\operatorname{grad}\sigma} \operatorname{grad}\sigma, Y\big) = -g\big(\operatorname{grad}\sigma, \nabla_{\operatorname{grad}\sigma} Y\big)$. On the other hand $g\big(\nabla_{\operatorname{grad}\sigma} \operatorname{grad}\sigma, Y\big) = \operatorname{Hess}^g \sigma(\operatorname{grad}\sigma, Y) = g\big(\operatorname{grad}\sigma, \nabla_{\operatorname{grad}\sigma} Y\big)$ and consequently $g\big(\nabla_{\operatorname{grad}\sigma} \operatorname{grad}\sigma, Y\big) = 0$ for such tangent vector fields. By definition of the metric one also has $g\big(\nabla_{\operatorname{grad}\sigma} \operatorname{grad}\sigma, \operatorname{grad}\sigma\big) \equiv 0$. Together with the previous calculation this gives $\nabla_{\operatorname{grad}\sigma} \operatorname{grad}\sigma \equiv 0$.

Now consider the second equation, then

$$\begin{aligned} \left(\nabla_Z g_\sigma\right)(X,Y) &= -\left(\nabla_Z d\sigma^2\right)(X,Y) \\ &= -\operatorname{Hess}^\sigma(X,Z)d\sigma(Y) - \operatorname{Hess}^\sigma(Y,Z)d\sigma(X) \\ &= 0, \end{aligned}$$

since $\operatorname{grad}\sigma$ is the generic normal vector field to the slices. ∎

Lemma A.1.5. *Consider K to be a $(2,0)$ tensor field and X, Y, V, W to be vector fields on M, then it holds*

$$\mathcal{L}_X\mathcal{L}_Y K - \mathcal{L}_Y\mathcal{L}_X K = \mathcal{L}_{[X,Y]}K.$$

Proof:

$$\begin{aligned} &\left(\mathcal{L}_X\mathcal{L}_Y K\right)(V,W) - \left(\mathcal{L}_Y\mathcal{L}_X K\right)(V,W) \\ &\quad = [X,Y]\left(K(V,W)\right) \\ &\qquad - \nabla_X\left(K([Y,V],W) + K(V,[Y,W])\right) + \nabla_Y\left(K([X,V],W) + K(V,[X,W])\right) \\ &\qquad - \left(\mathcal{L}_Y K\right)([X,V],W) - \left(\mathcal{L}_Y K\right)(V,[X,W]) + \left(\mathcal{L}_X K\right)([Y,V],W) - \left(\mathcal{L}_X K\right)(V,[Y,W]) \\ &\quad = [X,Y]\left(K(V,W)\right) \\ &\qquad + K([Y,[X,V]],W) + K([X,V],[Y,W]) + K([Y,V],[X,W]) + K(V,[Y,[X,W]]) \\ &\qquad - K([X,[Y,V]],W) - K([Y,V],[X,W]) - K([X,V],[Y,W]) - K(V,[X,[Y,W]]) \\ &\quad = [X,Y]\left(K(V,W)\right) - K([V,[Y,X]],W) - K(V,[W,[Y,X]]) \end{aligned}$$

∎

Lemma A.1.6. *Let $S^{p,q}$ be the pseudosphere defined in section 2.1 and consider the maps*

$$\begin{array}{cccc} \imath: & \mathbb{R}^{p,q} & \longrightarrow & S^{p,q} \\ & \hat{x} & \longmapsto & \pi_{S^{n+1}_{\sqrt{2}}}\left(1+\langle\hat{x},\hat{x}\rangle_{p,q}, 2\hat{x}, 1-\langle\hat{x},\hat{x}\rangle_{p,q}\right). \end{array}$$

and with $\sigma(x) = x^0 + x^{n+1}$

$$\begin{array}{cccc} \imath^{-1}: & \mathbb{R}^{p+1,q+1}\setminus\sigma^{-1}(0) & \longrightarrow & \mathbb{R}^{p,q} \\ & x=(x_0,\hat{x},x_{n+1}) & \longmapsto & \frac{\hat{x}}{\sigma(x)}. \end{array}$$

Then

$$\imath^{-1}\circ\imath = \mathrm{id}_{\mathbb{R}^{p,q}} \qquad\qquad \imath\circ\imath^{-1} =|_{\imath(\mathbb{R}^{p,q})} \mathrm{id}_{S^{p,q}}.$$

Proof: Consider $x = \left(x^0, \hat{x}, x^{n+1}\right) \in S^{p,q}$, then

$$\begin{aligned} \imath\circ\imath^{-1}\left(x^0,\hat{x},x^{n+1}\right) &= \pi_{S^{n+1}_{\sqrt{2}}}\left(1+\frac{\langle\hat{x},\hat{x}\rangle_{p,q}}{(x^0+x^{n+1})^2}, \frac{2\hat{x}}{x^0+x^{n+1}}, 1-\frac{\langle\hat{x},\hat{x}\rangle_{p,q}}{(x^0+x^{n+1})^2}\right) \\ &= \pi_{S^{n+1}_{\sqrt{2}}}\left(\frac{2\sigma(x)x^0+\langle x,x\rangle_{p+1,q+1}}{\sigma^2(x)}, \frac{2\hat{x}}{\sigma(x)}, \frac{2\sigma(x)x^{n+1}-\langle x,x\rangle_{p+1,q+1}}{\sigma(x)^2}\right) \\ &= \pi_{S^{n+1}_{\sqrt{2}}}\left(\sigma(x)\left(x^0+0,\hat{x},x^{n+1}-0\right)\right) \\ &= \left(x^0,\hat{x},x^{n+1}\right) \end{aligned}$$

On the other hand for $\hat{x}\in\mathbb{R}^{p,q}$ one finds

$$\begin{aligned} \imath^{-1}\circ\imath(\hat{x}) &= \imath^{-1}\left(\frac{1}{N(\hat{x})}\left(1+\langle\hat{x},\hat{x}\rangle_{p,q}, 2\hat{x}, 1-\langle\hat{x},\hat{x}\rangle_{p,q}\right)\right) \\ &= \frac{2\hat{x}}{2}. \end{aligned}$$

■

Lemma 4.1.4 *Let ∇ and D be two torsion-free connections on M and $\mathcal{M} = \nabla - D$ the potential. Assume both connections to be canonically extended to act on arbitrary tensor fields T by Equation (1.1). Let X be a vector field on M. Then the covariant derivative of T can be expressed by using the Ricci product, as*

$$\nabla_X T = D_X T + \mathcal{M}(X)\cdot T, \tag{A.10}$$

where $\mathcal{M}(X) := \mathcal{M}(X,\cdot)$ is a $(1,1)$-tensor field.

Proof: The term $\mathcal{M}(X)$ in the previous lemma is well defined, since $\mathcal{M}$ is symmetric and the remaining $(1,1)$-tensor $\mathcal{M}(X)$ does not depend on the slot that has been dualised. Then one has

$$\left(\nabla_X T - D_X T\right)(\theta_1,\ldots,\theta_k) = -\sum_i T\left(\ldots,\nabla_X\theta_i - D_X\theta_i,\ldots\right).$$

Now for vector fields one simply has $\nabla_X\theta_i - D_X\theta_i = \mathcal{M}(X,\theta_i) = \mathcal{M}(X)\cdot\theta_i$, while on forms $\left(\nabla_X\theta_i - D_X\theta_i\right)(Y) = -\theta_i\left(\mathcal{M}(X,Y)\right) = \left(\mathcal{M}(X)\cdot\theta_i\right)(Y)$ and the claimed formula follows immediately. ■

Lemma 5.1.11 *For $p\in\Sigma_d$ there are convex sets $\mathfrak{U}\subset\mathfrak{K}\subset\tilde{\mathfrak{U}}\subset T_pM$ containing the origin 0 such that $\mathfrak{K}$ is compact and $\mathfrak{U}$, $\tilde{\mathfrak{U}}$ are stable neighbourhoods with respect to* $\operatorname{grad}\sigma$ *at p.*

Proof: Choosing normal coordinates (U,φ) and using Proposition 1.1.18 we find analogous to the proof of Corollary 5.1.9 an attracting or repelling ball $B_r(0)\subset\varphi(U)$ of $0=\varphi(p)$ with respect to $\varphi_*\operatorname{grad}\sigma$. Moreover due to Proposition 1.1.18, every ball $B_{\tilde{r}}$ with $\tilde{r}<r$ is attracting or repelling. We choose $0<r_1<r_2<r$ and define

$$\mathfrak{U} := \left(\varphi\circ\exp_p\right)^{-1}\left(B_{r_1}(0)\right) \quad \mathfrak{K} := \left(\varphi\circ\exp_p\right)^{-1}\left(\overline{B_{r_2}(0)}\right) \quad \tilde{\mathfrak{U}} := \left(\varphi\circ\exp_p\right)^{-1}\left(B_r(0)\right).$$

These sets have the desired properties by definition. ■

Lemma 5.3.17 *Let $S^n = \bigcup_{i=1}^{k} B_i$ be a finite cover of open balls B_i. Then for each $i \in \{1, \dots, m\}$ there is a smaller ball $\tilde{B}_i \subset B_i$ and smooth maps $f_i : S^n \to [0,1]$ and $F_i : S^n \to [0,1]$ such that*

$$S^n = B_1 \cup \cdots \cup B_{i-1} \cup \tilde{B}_i \cup B_{i+1} \cup \cdots \cup B_k$$

still is an open cover, $\overline{\text{supp}(F_i)} \subset B_i$ is compact and

$$\text{supp}(f_i) \subset \text{supp}(F_i) \qquad f_i|_{\tilde{B}_i} \equiv 1 \qquad F_i|_{\text{supp}(f_i)} \equiv 1. \tag{A.11}$$

Proof: Existence of $\tilde{B}_i$ is a consequence of finiteness of the covering. Fix B_i, then the union of the remaining balls gives an open neighbourhood of the boundary ∂B_i. The complement of the union is a subset of B_i, since we started with a covering. Hence the radius r of B_i can be reduced to get a smaller ball $\tilde{B}_i$ with radius $\tilde{r}$, which has the same properties. The maps f and F then are be defined as smooth functions of the radius of balls having the same centre as B_i. They have to be identically 1 for radii smaller than $\tilde{r}$ and have to vanish for radii greater than r. After having constructed f with this properties, F can be constructed in the same way by replacing $\tilde{r}$ by a greater radius (but smaller than r), where f already started vanishing. ■

B HYPERSURFACE RICCI

The notation of section 2.2 will be used in the following. Vanishing Lie derivatives of the induced Ricci tensor $\mathrm{Ric}[g_\sigma]$ will be shown after introducing a pullback derivative operator on horizontal tensors.

B.1 PULLBACK DERIVATIVE OPERATOR ON HORIZONTAL TENSORS

Let Σ be a smooth manifold. On $M = \mathbb{R} \times \Sigma$ consider the bundle of horizontal (p, q) tensors $Th^{p,q}M$, i.e. tensors that are annihilated by ∂_σ or $d\sigma$, where σ is the parameter of the first component of the Cartesian product. The set of smooth sections in this bundle will be denoted $\mathcal{T}h^{p,q}M$. Let further $\imath_s : \Sigma \to \{s\} \times \Sigma$ be the natural inclusion and let $\pi : M \to \Sigma$ be the projection.

Definition B.1.1. Let $D : \mathcal{T}^{p,q}\Sigma \to \Gamma(T^*\Sigma) \otimes \mathcal{T}^{p,q}\Sigma$ be a torsion-free connection on Σ. Then D induces a pullback derivative operator $\mathcal{D} : \mathcal{T}h^{p,q}M \to \mathcal{T}h^{p+1,q}M$ on horizontal tensors by

$$(\mathcal{D}T)_{(s,x)} := (\pi^* D(\imath_s^* T))_{(s,x)} \,.$$

The pullback $\imath_s^* T$ of contravariant parts of tensors always is meant to imply the projection to the $T\Sigma$ component, e.g. $(\imath_s^* X)_x = \pi_2(X_{\imath_s(x)}) = d\pi_{(s,x)}(X)$. On the other hand π^* denotes the lift of tensors on $T^{p,q}\Sigma$ to horizontal tensors on $Th^{p,q}M$. For a vector $V \in T_x\Sigma$ this also can be written as $(\pi^* V)_{(s,x)} = (d\imath_s)_x(V)$.

In the notation of the last definition the composition $(\pi^* \imath_s^* T)_{(s,x)}$ at (s, x) is the projection of a tensor to its horizontal part and will be denoted with $T^{\|}$. In particular $\imath_s^* V = \imath_s^* V^{\|}$ for a vector field V on M.

Lemma B.1.2. *Let $T \in \mathcal{T}h^{p,0}M$ and X, Y_i vector fields on M then*

$$(\mathcal{D}T)(X, Y_1, \dots Y_p) = X^{\|}\left(T(Y_1, \dots, Y_n)\right) - T\left(\mathcal{D}_X Y_1^{\|}, Y_2, \dots\right) - \dots - T\left(\dots, Y_{p-1}, \mathcal{D}_X Y_p^{\|}\right).$$

Proof: One can use $(\imath_s^* V)_x = d\pi_{(s,x)}(V)$ for vector fields V on M and then gets

$$\begin{aligned}
&(\mathcal{D}T)_{(s,x)}(X, Y_1, \dots Y_p) \\
&\quad = (D(\imath_s^* T))_x \left(d\pi_{(s,x)}(X), d\pi_{(s,x)}(Y_1), \dots, d\pi_{(s,x)}(Y_p)\right) \\
&\quad = (\imath_s^* X)_x \left(\imath_s^* T\left((\imath_s^* Y_1, \dots, \imath_s^* Y_n\right)\right) \\
&\qquad - \sum_{i=1,..,p} (\imath_s^* T)_x \left(\dots, d\pi_{(s,x)}(Y_{i-1}), D_{d\pi_{(s,x)}(X)} \imath_s^* Y_i, d\pi_{(s,x)}(Y_{i+1}), \dots\right) \\
&\quad = X^{\|}_{(s,x)}\left(T(Y_1, \dots, Y_n)\right) \\
&\qquad - \sum_{i=1,..,p} (\imath_s^* T)_x \left(\dots, d\pi_{(s,x)}(Y_{i-1}), (D\imath_s^* Y_i)_{\pi(s,x)}\left(d\pi_{(s,x)}(X)\right), d\pi_{(s,x)}(Y_{i+1}), \dots\right).
\end{aligned}$$

■

Lemma B.1.3. *Let $N = \partial_\sigma$, Y an arbitrary vector field on M, Z a horizontal vector field and T a horizontal $(p, 0)$-tensor on M. The Lie derivative $\mathcal{L}_N$ acting on horizontal tensors gives horizontal tensors and hence one has* *(i)* $[N, \mathcal{D}_Y Z] = \mathcal{D}_{[N,Y]} Z + \mathcal{D}_Y [N, Z]$ *and*

(ii) $\mathcal{L}_N \mathcal{D} T = \mathcal{D} \mathcal{L}_N T$.

Proof: The first claim is a result of Schwarz theorem in coordinates or may be proven independent of a choice of coordinates as follows. Consider the flow Φ of N. It is provided by $\Phi^t(s,x) = (s+t,x)$. Then the following formulas hold

$$(\mathcal{D}_Y Z)_{(s,x)} = d\Phi^s_{(0,x)} \circ (d\imath_0)_x \left(D_{d\pi_{(s,x)}(Y)} \imath_s^* Z \right) \qquad \begin{aligned} d\pi_{(s+t,x)}(Y) &= d\pi_{(s,x)} \circ d\Phi^{-t}_{\Phi^t(s,x)}(Y) \\ (\imath_{s+t}^* Z)_x &= \left(\imath_s^* \Phi_*^{-t} Z\right)_x , \end{aligned}$$

where again $\imath_s^* Z$ implies projection to the $T\Sigma$ component. As a consequence

$$\begin{aligned} (\mathcal{L}_N(\mathcal{D}_Y Z))_{(s,x)} &= \left.\frac{d}{dt}\right|_{t=0} d\Phi^{-t}_{(s+t,x)} \left((\mathcal{D}_Y Z)_{(s+t,x)} \right) \\ &= \left.\frac{d}{dt}\right|_{t=0} d\Phi^{-t}_{(s+t,x)} \circ d\Phi^{s+t}_{(0,x)} \circ (d\imath_0)_x \left(D_{d\pi_{(s,x)} \circ d\Phi^{-t}_{\Phi^t(s,x)}(Y)} \imath_{s+t}^* Z \right) \\ &= \left.\frac{d}{dt}\right|_{t=0} d\Phi^{s}_{(0,x)} \circ (d\imath_0)_x \left(D_{d\pi_{(s,x)} \circ d\Phi^{-t}_{\Phi^t(s,x)}(Y)} \imath_s^* \Phi_*^{-t} Z \right) . \end{aligned}$$

Parameter derivation of $\imath_s^* \Phi_*^{-t} Z$ gives

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} \left(\imath_s^* \Phi_*^{-t} Z\right)_x &= \pi^{T_x\Sigma} \left(\left.\frac{d}{dt}\right|_{t=0} \left(\Phi_*^{-t} Z\right)_{(s,x)} \right) \\ &= \pi^{T_x\Sigma} \left(\left.\frac{d}{dt}\right|_{t=0} d\Phi^{-t}_{\Phi^t(s,x)} \left(Z_{\Phi^t(s,x)} \right) \right) \\ &= \pi^{T_x\Sigma} \left((\mathcal{L}_N Z)_{(s,x)} \right) \\ &= (\imath_s^* \mathcal{L}_N Z)_x \end{aligned}$$

and $\mathbb{R}$-linearity of the connection then provides the first claim since

$$\begin{aligned} (\mathcal{L}_N(\mathcal{D}_Y Z))_{(s,x)} &= d\Phi^s_{(0,x)} \circ (d\imath_0)_x \left(D_{d\pi_{(s,x)}(\mathcal{L}_N Y)} \imath_s^* Z + D_{d\pi_{(s,x)}(Y)} \imath_s^* \mathcal{L}_N Z \right) \\ &= (d\imath_s)_x \left(D_{d\pi_{(s,x)}(\mathcal{L}_N Y)} \imath_s^* Z + D_{d\pi_{(s,x)}(Y)} \imath_s^* \mathcal{L}_N Z \right) \\ &= (\pi^* D \imath_s^* Z)_{(s,x)} (\mathcal{L}_N Y) + (\pi^* D \imath_s^* \mathcal{L}_N Z)_{(s,x)} (Y) . \end{aligned}$$

The second claim now can be proven as follows,

$$\begin{aligned} &(\mathcal{L}_N \mathcal{D} T)_{(s,x)} (X, Y_1, \ldots, Y_p) \\ &\quad = N \left((\mathcal{D}T)_{(s,x)} (X, Y_1, \ldots, Y_p) \right) \\ &\qquad - (\mathcal{D}T)_{(s,x)} ([N,X], Y_1, \ldots, Y_p) - \cdots - (\mathcal{D}T)_{(s,x)} (X, \ldots, [X, Y_i], \ldots) - \ldots \\ &\quad = N \left(X^{\|}(T(Y_1, \ldots, Y_p)) - \cdots - T\left(\ldots, \mathcal{D}_X Y_i^{\|}, \ldots \right) - \ldots \right) \\ &\qquad - [N,X]^{\|}(T(Y_1, \ldots, Y_p)) + \cdots + T(\ldots, \mathcal{D}_{[N,X]} Y_i^{\|}, \ldots) + \ldots \\ &\qquad \vdots \\ &\qquad - X^{\|}(T(\ldots, Y_{i-1}, [N, Y_i], Y_{i+1,\ldots})) + \cdots + T\left(\ldots, Y_{i-1}, \mathcal{D}_X [N, Y_i]^{\|}, Y_{i+1,\ldots} \right) + \ldots \\ &\qquad \vdots \\ &\quad = (\mathcal{D}(\mathcal{L}_N T))(X, Y_1, \ldots, Y_p) \\ &\qquad + \cdots + T\left(\ldots, Y_{i-1}, \mathcal{D}_{[N, X^{\|}]} Y_i^{\|} + \mathcal{D}_{X^{\|}} [N, Y_i^{\|}] - \left[N, \mathcal{D}_{X^{\|}} Y_i^{\|} \right], Y_{i+1}, \ldots \right) + \ldots \\ &\quad \overset{(i)}{=} (\mathcal{D}(\mathcal{L}_N T))(X, Y_1, \ldots, Y_p). \end{aligned}$$

■

B.2 VANISHING OF ODD DERIVATIVES

Lemma B.2.1. *Consider the metric* $g = d\sigma^2 + g_\sigma$ *on the product manifold* $I \times \Sigma$ *for some interval* $I \subset \mathbb{R}$ *that contains the origin. If* $g^{(j)} =|_{\Sigma_0} 0$ *for all odd* $j \leq k$ *for some* $k \in \mathbb{N}$*, then* $\mathrm{Ric}[g_\sigma]^{(j)} =|_{\Sigma_0} 0$ *for all odd* $j \leq k$ *as well.*

Proof: First, vanishing of odd derivatives ${g^*}^{(j)}$ of the dual metric g^* up to order k is provided by an inductive argument (see section 2.2.1). Now assume $D : \mathcal{T}^{p,q}\Sigma \to \Gamma(T^*\Sigma) \otimes \mathcal{T}^{p,q}\Sigma$ to be a torsion-free connection on Σ. By pullback this naturally induces the above derivative operator $\mathcal{D} : \Gamma\left(Th^{p,q}M\right) \to \Gamma(Th^*\Sigma) \otimes \Gamma\left(Th^{p,q}M\right)\Sigma$ on horizontal tensors on M. This derivative has the property to commute with the Lie-derivative $\mathcal{L}_{\mathrm{grad}\,\sigma}$. Now $\mathrm{Ric}[g_\sigma]$ can be written as $\mathrm{Ric}[g_\sigma] = L[g^*_\sigma, \dots, g^*_\sigma, g_\sigma, \dots, g_\sigma, \mathcal{D}g_\sigma, \dots, \mathcal{D}g_\sigma, \mathcal{D}\mathcal{D}g_\sigma, \dots, \mathcal{D}\mathcal{D}g_\sigma]$ (equ. (4.8)), where g^*_σ is the dual of g_σ along Σ_σ and $L[A_1, \dots, A_m]$ is a linear map on horizontal tensors with values in horizontal $(2,0)$-tensors that depends only on contractions of tensor products of the A_i. As the contraction commutes with the Lie derivative this in particular implies $\mathcal{L}_{\mathrm{grad}\,\sigma} L[A_1, \dots, A_m] = L[\mathcal{L}_{\mathrm{grad}\,\sigma} A_1, A_2, \dots] + \dots + L[\dots, A_{m-1}, \mathcal{L}_{\mathrm{grad}\,\sigma} A_m]$, such that

$$\mathrm{Ric}[g_\sigma]^{(k)} = \sum_{|J|=k} a_J L\left[{g^*_\sigma}^{(j_1)}, \dots, \mathcal{D}\mathcal{D}g_\sigma^{(j_m)}\right],$$

with multinomial coefficients a_J. In case where k is odd, at least one of the j_i has to be odd as well, such that $\mathrm{Ric}[g_\sigma]^{(k)}$ is a sum of zeros along Σ_0 due to the requirements of the lemma. ■

C ENERGY INEQUALITIES

The energy inequality used to provide local vanishing of the wave gauge vector works as follows (compare [Eva98, Theorem 12.3]). Here only the local formulation of the corresponding problem in coordinates is sketched, where initial data are given on a disc or on a cone. One considers a linear wavelike equation in flat $\mathbb{R}^n$ of the following type.

$$\Box u = f(t, x, u, \operatorname{grad}^{\mathbb{R}^{n-1}} u, \dot{u}), \tag{C.1}$$

where $\Box := -\partial_t^2 + \sum_{i=1,\dots,n-1} \partial_i^2$ is the Minkowski Laplace on functions and f is a sufficiently smooth map linear in u, $\operatorname{grad}^{\mathbb{R}^{n-1}} u$ and $\dot{u}$. $\operatorname{grad}^{\mathbb{R}^{n-1}} u$ represents the gradient along $\{t = const\}$-slices in $\mathbb{R}^n$. Using linearity in the last arguments, on a compact set $K \subset \mathbb{R}^n$ there is a constant C such that $\left| f(t, x, u, \operatorname{grad}^{\mathbb{R}^{n-1}} u, \dot{u}) \right| \leq C \left(\| \operatorname{grad}^{\mathbb{R}^{n-1}} u \| + |\dot{u}| + |u| \right)$ for all $(t, x) \in K$. This inequality does not depend on the map u. Now the following definitions are needed. $B_x(r) \subset \mathbb{R}^{n-1}$ is the ball with origin x and radius r, $S_x(r) = \partial B_x(r)$ is the corresponding sphere and $K^{\downarrow}(t_0, x_0) = \left\{ (\tilde{t}, \tilde{x}) \in [0, t_0] \times \mathbb{R}^{n-1} \mid \|x - x_0\|_{n-1} \leq t_0 - t \right\}$ is the backward cone of (t_0, x_0). To keep the formulas short from now on the notation $\operatorname{grad} := \operatorname{grad}^{\mathbb{R}^{n-1}}$ will be used. Since there is no need for explicitly using the $\mathbb{R}^n$-gradient, this should not lead to a misunderstanding.

Let u be a sufficiently smooth solution to the above equation in a neighbourhood of the the domain $K^{\downarrow}$. The energy of u for $t \in (0, t_0)$ then is defined by

$$e(t) := \frac{1}{2} \int_{B_{x_0}(t_0 - t)} \left(\dot{u}^2 + \langle \operatorname{grad} u, \operatorname{grad} u \rangle_{n-1} + u^2 \right) dx,$$

where dx is the usual Lebesgue measure on $\mathbb{R}^{n-1}$. By differentiating in t, using a Leibniz integral rule (e.g. [Fla73]) and by use of the notation $\dot{f} = \partial_t f$ for t-derivatives one obtains

$$\dot{e}(t) = \int_{B_{x_0}(t_0 - t)} \left(\dot{u}\ddot{u} + \langle \operatorname{grad} u, \operatorname{grad} \dot{u} \rangle_{n-1} + u\dot{u} \right) dx - \frac{1}{2} \int_{S^{n-2}_{x_0}(t_0 - t)} \left(\dot{u}^2 + \langle \operatorname{grad} u, \operatorname{grad} u \rangle_{n-1} + u^2 \right) dS,$$

where dS is the surface measure on the sphere $\partial B_{x_0}(t_0 - t) = S^{n-2}_{x_0}(t_0 - t)$. The second term appears due to the constant rate of change of the radius of $B_{x_0}(t_0 - t)$. Using Stokes theorem on $\operatorname{div}(\dot{u} \operatorname{grad} u)$ and having u to be a solution to Equation (C.1) one gets

$$\begin{aligned}
\dot{e}(t) = & \int_{B_{x_0}(t_0 - t)} \Big(\dot{u} \left(f(x, u, D_x u, \dot{u}) + u \right) \Big) dx \\
& + \frac{1}{2} \int_{S^{n-2}_{x_0}(t_0 - t)} \left(2 \langle e_x, \operatorname{grad} u \rangle_{n-1} \dot{u} - \dot{u}^2 - \langle \operatorname{grad} u, \operatorname{grad} u \rangle_{n-1} - u^2 \right) dS \\
= & \int_{B_{x_0}(t_0 - t)} \Big(\dot{u} \left(f(x, u, D_x u, \dot{u}) + u \right) \Big) dx - \frac{1}{2} \int_{S^{n-2}_{x_0}(t_0 - t)} \left(\langle \dot{u} e_x - \operatorname{grad} u, \dot{u} e_x - \operatorname{grad} u \rangle_{n-1} + u^2 \right) dS \\
\leq & \int_{B_{x_0}(t_0 - t)} \Big(\dot{u} \left(f(x, u, D_x u, \dot{u}) + u \right) \Big) dx \\
\leq & \int_{B_{x_0}(t_0 - t)} \Big(\dot{u} u + C \left(\dot{u} \| \operatorname{grad} u \|_{n-1} + \dot{u} |\dot{u}| + \dot{u} |u| \right) \Big) dx.
\end{aligned}$$

where the last equation is due to the linearity of f on the compact part of the cone where $0 \leq t \leq 1$. The vector $e_x = x \in \mathbb{R}^{n-1}$ in the last equation is the normal vector of the sphere at x. Now using the inequality $2ab \leq a^2 + b^2$ on the last three terms, one finds a new constant $\tilde{C}$ such that

$$\begin{aligned}\dot{e}(t) &\leq \tilde{C} \int\limits_{B_{x_0}(t_0-t)} \Big(\dot{u}^2 + \|\operatorname{grad} u\|^2_{n-1} + u^2\Big) dx \\ &= \tilde{C} e(t).\end{aligned}$$

Consequently a solution u to the linear PDE (C.1) with $u(x,0), \dot{u}(x,0) = 0$ for all $x \in B_{x_0}(t_0)$ will have vanishing energy $e(0) = 0$ and by Gronwall's inequality will have vanishing energy $e(t) = 0$ for all $t \in [0, t_0]$. Hence u vanishes at the causal cone $K^{\downarrow}(t_0, x_0)$. The latter is also called *domain of dependence* corresponding to $\{0\} \times B_{x_0}(t_0)$.

The reasoning is quite similar in case where the initial data are given on the boundary of a future-directed causal cone in the origin $K^{\uparrow} = \big\{(\tilde{t}, \tilde{x}) \in [0, t_0] \times \mathbb{R}^{n-1} \,|\, \|x\|_{n-1} \leq t\big\}$ (compare ([Fri75]) for a generalised treatment). The energy is defined in the same way with a little modification for the radius of the ball, since the vertex here is at $(t, x) = 0$. In particular

$$e(t) := \frac{1}{2} \int\limits_{B(t)} \Big(\dot{u}^2 + \langle \operatorname{grad} u, \operatorname{grad} u \rangle_{n-1} + u^2\Big) dx,$$

where the ball with radius t is centred at the origin of $\mathbb{R}^{n-1}$. Let u be a sufficiently smooth solution to (C.1) with vanishing initial data on the cone, i.e. $u(t,x) = 0$ for all (t,x) such that $t = \|x\|_{n-1}$ and $t \in [0, t_0]$. First this implies $du(X) = X(u) = 0$ for all tangent vector fields on the cone $C = \{(t,x) \in \mathbb{R} \,|\, t = \|x\|_{n-1}, t \in [0, t_0]\}$. Hence by Proposition 1.2.6 $du_0 = 0$ and therefore $e(0) = 0$. Moreover, since u is constant along the cone its gradient $(\dot{u}, \operatorname{grad} u)$ is normal to the cone C with respect to the inner product $\langle \cdot, \cdot \rangle_n$ and hence collinear to $(-t, x)$. Along the cone C one has $t = \|x\|_{n-1}$ and hence $\dot{u} = -\langle e_x, \operatorname{grad} u \rangle$, where $e_x = \frac{x}{\|x\|}$ is defined away from $x = 0$ and $\dot{u}^2 = \langle \operatorname{grad} u, \operatorname{grad} u \rangle_{n-1}$. Differentiating the energy in t yields

$$\dot{e}(t) = \int\limits_{B(t)} \Big(\dot{u}\ddot{u} + \langle \operatorname{grad} u, \operatorname{grad} \dot{u} \rangle_{n-1} + u\dot{u}\Big) dx + \frac{1}{2} \int\limits_{\partial B(t) = S^{n-2}(t)} \Big(\dot{u}^2 + \langle \operatorname{grad} u, \operatorname{grad} u \rangle_{n-1} + u^2\Big) dS$$

By using Stokes theorem and vanishing of u along the cone, one finds

$$\begin{aligned}\dot{e}(t) &= \int\limits_{B(t)} \Big(\dot{u}\,(f + u)\Big) dx + \frac{1}{2} \int\limits_{S^{n-2}(t)} \Big(2\langle e_x, \operatorname{grad} u \rangle \dot{u} + \dot{u}^2 + \langle \operatorname{grad} u, \operatorname{grad} u \rangle_{n-1}\Big) dS \\ &= \int\limits_{B(t)} \Big(\dot{u}\,(f + u)\Big) dx + \frac{1}{2} \int\limits_{S^{n-2}(t)} \Big(-\dot{u}^2 + \langle \operatorname{grad} u, \operatorname{grad} u \rangle_{n-1}\Big) dS \\ &= \int\limits_{B(t)} \Big(\dot{u}\,(f + u)\Big) dx.\end{aligned}$$

The final estimate works as before such that $e(0) = 0$ implies $e \equiv 0$ inside the causal cone $K^{\uparrow}$ if u vanishes on its boundary. Hence $u \equiv 0$ on $K^{\uparrow}$.

D DIFFERENTIAL-ALGEBRAIC EQUATIONS

Let $I \subset \mathbb{R}$ be an interval and $U \subset \mathbb{R}^k$ be an parameter set. Consider a map $X : I \times U \to \mathrm{Gl}(n-2, \mathbb{R})$ and let X^{-1} be the inverse matrix. Derivatives with respect to the first parameter are denoted by a dot, i.e. $\dot{X} = \frac{\partial}{\partial t}$.

Lemma D.0.2. *X is a smooth solution on $I \times U$ to the first-order differential-algebraic system of equations*

$$(n-2)\dot{X}_{ab} - X_{ab} \sum_{i,j=1}^{n-2} \left(X^{-1}\right)_{ij} \dot{X}_{ij} = 0 \tag{D.1}$$

if and only if it is of the form $X(t,p) = f(t,p)A(p)$ with $f : I \times U \to \mathbb{R}$ and $A : U \to \mathrm{Gl}_{n-2}(\mathbb{R})$.

Proof: First let X be of the claimed form, then $X^{-1} = {}^1/_f A^{-1}$ and Equation (D.1) is trivially fulfilled as

$$\begin{aligned}(n-2)\dot{f}A_{ab} - fA_{ab} \sum_{i,j=1}^{n-2} \frac{1}{f} \left(A^{-1}\right)_{ij} \dot{f}A_{ij} &= (n-2)\dot{f}A_{ab} - \dot{f}A_{ab} \sum_{i,j=1}^{n-2} \left(A^{-1}\right)_{ij} A_{ij} \\ &= 0.\end{aligned}$$

For the converse consider the X_{ab} to be the components of an $(n-2)^2$-dimensional vector, which is labelled by two indices. If the two labels are replaced by just one, then Equation (D.1) may also be written as $(n-2)\dot{X}_\mu - X_\mu \sum_{\nu=1}^{n-2} \left(X^{-1}\right)_\nu \dot{X}_\nu = 0$ or even simpler as

$$\left((n-2)\mathbb{1} - X^t \cdot X^{-1}\right) \dot{X}^t = 0,$$

where X^{-1} refers to the vector that is generated from the matrix with the same symbol. Written in this form it is obvious that for any solution X of the differential equation, the vector $\dot{X}$ must be an eigenvector of the matrix $M := X^t X \in \mathrm{Mat}_{(n-2)^2}(\mathbb{R})$ with eigenvalue $(n-2)$. As the only eigenvector of the rank 1 matrix M is X, $\dot{X}$ must be proportional to X for all $t \in I$. Hence X fulfils a linear ordinary differential equation of type $\dot{X} = \kappa X$, where $\kappa : I \times U \to \mathbb{R}$ is a smooth map. The only solutions to such an equation are of the claimed form. ■

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SELBSTSTÄNDIGKEITS- UND EINVERSTÄNDNISERKLÄRUNG

Selbstständigkeitserklärung

Ich erkläre, dass ich die Dissertation selbständig und nur unter Verwendung der von mir gemäß §7 Abs. 3 der Promotionsordnung der Mathematisch-Naturwissenschaftlichen Fakultät, veröffentlicht im Amtlichen Mitteilungsblatt der Humboldt-Universität zu Berlin Nr. 126/2014 am 18.11.2014 angegebenen Hilfsmittel angefertigt habe. Ich habe mich nicht anderwärts um einem Doktorgrad im Promotionsfach Mathematik beworben und besitze keinen Doktorgrad im Promotionsfach Mathematik.
Die Promotionsordnung der Mathematisch-Naturwissenschaftlichen Fakultät, veröffentlicht im Amtlichen Mitteilungsblatt der Humboldt-Universität zu Berlin Nr. 126/2014 am 18.11.2014 habe ich zur Kenntnis genommen.

Berlin, den 22.10.2015 Peter Schemel